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Mathematics Foundation Course Unit 29

Complex Numbers II

The Open University

Mathematics Foundation Course Unit 29

COMPLEX NUMBERS II

Prepared by the Mathematics Foundation Course Team

Correspondence Text 29

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1.2

Contents

Objectives

The principal objective of this unit is to introduce functions from the set of complex numbers to itself.

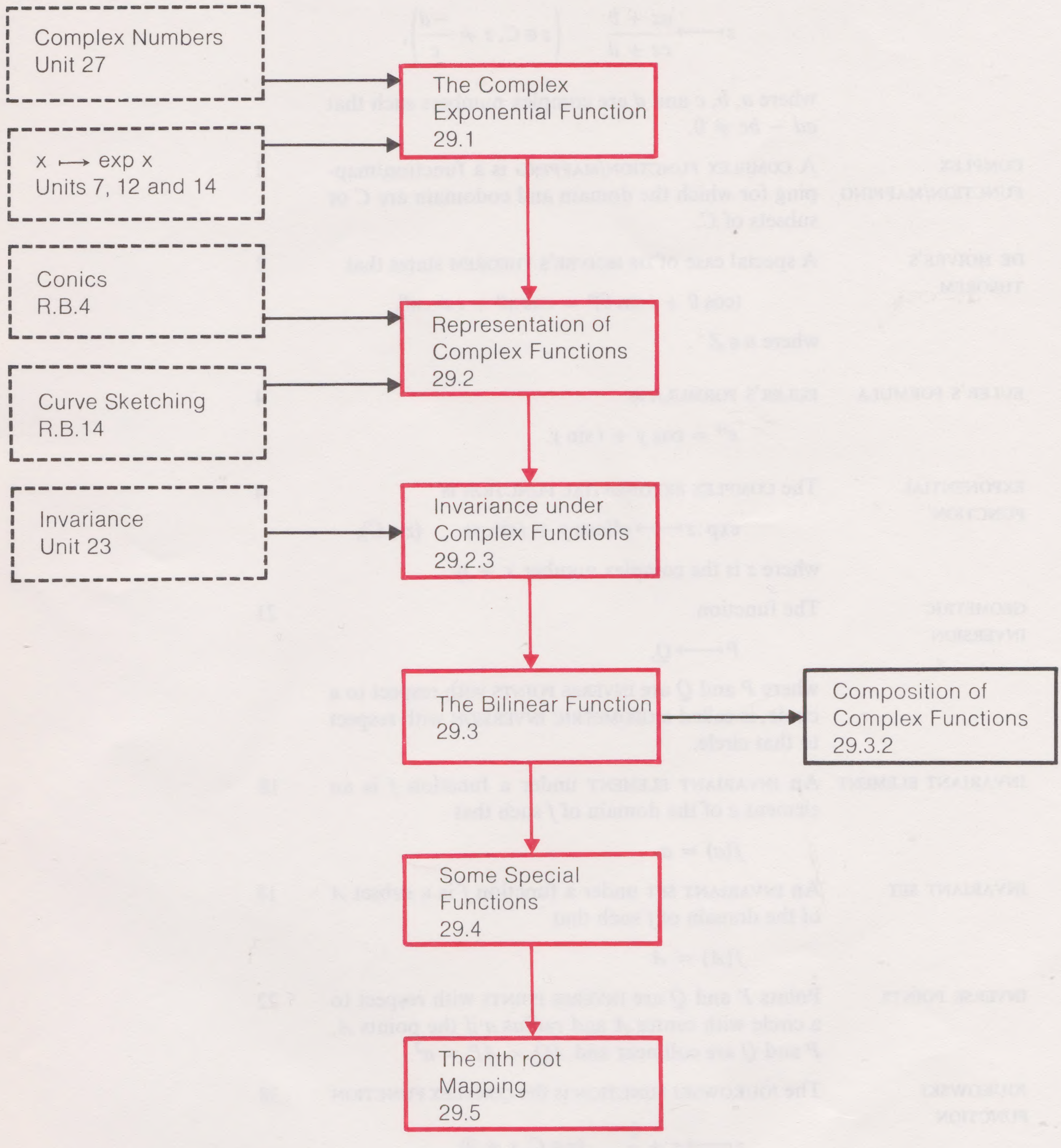
After working through this unit you should be able to:

(i) define the complex exponential function and be familiar with some of its elementary properties;
(ii) find the image of a given set in the complex plane under a given elementary function;
(iii) prove that the set of all circles and straight lines is invariant under a bilinear function;
(iv) find invariant points and sets under a given elementary function;
(v) express a complex function as a composition of elementary functions;
(vi) find the roots of the equation $z^n = 1$;
(vii) solve a given equation involving nth roots.

Note

Before working through this correspondence text, make sure you have read the general introduction to the mathematics course in the Study Guide, as this explains the philosophy underlying the whole course. You should also be familiar with the section which explains how a text is constructed and the meanings attached to the stars and other symbols in the margin, as this will help you to find your way through the text.

Structural Diagram



Glossary

Page

Terms which are defined in this glossary are printed in CAPITALS.

		Page
BILINEAR FUNCTION	A BILINEAR FUNCTION is a COMPLEX FUNCTION of the form $$z \longmapsto \frac{az+b}{cz+d} \qquad \left(z \in C, z \neq \frac{-d}{c}\right),$$ where a, b, c and d are complex numbers such that $ad - bc \neq 0$.	21
COMPLEX FUNCTION/MAPPING	A COMPLEX FUNCTION/MAPPING is a function/mapping for which the domain and codomain are C or subsets of C.	1
DE MOIVRE'S THEOREM	A special case of DE MOIVRE'S THEOREM states that $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$ where $n \in Z^+$.	2
EULER'S FORMULA	EULER'S FORMULA is $$e^{iy} = \cos y + i\sin y.$$	4
EXPONENTIAL FUNCTION	The COMPLEX EXPONENTIAL FUNCTION IS $$\exp : z \longmapsto e^x(\cos y + i\sin y) \qquad (z \in C),$$ where z is the complex number $x + iy$.	4
GEOMETRIC INVERSION	The function $$P \longmapsto Q,$$ where P and Q are INVERSE POINTS with respect to a circle, is called a GEOMETRIC INVERSION with respect to that circle.	21
INVARIANT ELEMENT	An INVARIANT ELEMENT under a function f is an element a of the domain of f such that $$f(a) = a.$$	18
INVARIANT SET	An INVARIANT SET under a function f is a subset A of the domain of f such that $$f(A) = A$$	18
INVERSE POINTS	Points P and Q are INVERSE POINTS with respect to a circle with centre A and radius a if the points A, P and Q are collinear and $AQ \times AP = a^2$	22
JOUKOWSKI FUNCTION	The JOUKOWSKI FUNCTION is the COMPLEX FUNCTION $$z \longmapsto z + \frac{1}{z} \qquad (z \in C, z \neq 0).$$	38

Notation

Page

The symbols are presented in the order in which they appear in the text.

$\otimes$	The operation on the set of Cartesian co-ordinates which corresponds to multiplication on the set of polar co-ordinates.	1
C	The set of all complex numbers.	1
z	The complex number $x + iy$. The polar co-ordinates of z are usually written as (r, θ).	2
C_1	The set C without the zero element.	3
exp	The complex exponential function: $\exp: z \longmapsto e^x(\cos y + i \sin y) \quad (z \in C)$, where $z = x + iy$.	4
$\lvert z \rvert$	The modulus of z.	5
$\bar{z}$	The complex conjugate of z.	5
$w = u + iv$	The image of $z = x + iy$ under a particular complex mapping. The polar co-ordinates of w are usually written as (ρ, ϕ).	10
p	The many-one mapping which maps polar co-ordinates to the corresponding Cartesian co-ordinates.	47

Bibliography

For an introduction to complex numbers and complex functions, see

F. J. Budden, *Complex Numbers and Their Applications* (Longmans 1968).
This book discusses some of the applications of complex numbers; in particular, it discusses the Joukowski aerofoil, which is mentioned in this text and in the corresponding television programme.

A light and very brief introduction to complex numbers and the complex exponential function can be found in the paperback
W. W. Sawyer, *Mathematician's Delight* (Penguin Books 1943).

29.0 INTRODUCTION

29.0

Introduction

In this unit we shall take up the story of complex numbers from where we left it in *Unit 27, Complex Numbers I*. In that unit we were mainly concerned with building an algebraic structure, and we did this by introducing a further binary operation on the vector space of ordered pairs of real numbers. We called this binary operation "multiplication"; in terms of number pairs it was defined by

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2).$$

We introduced the notation $(x, y) = x + iy$, which is very useful because it enables us to work with this apparently complicated rule for multiplication without having to remember the above formula. With this notation we simply use the familiar rules of addition and multiplication, as in the algebra of real numbers: whenever we see i^2 we replace it by -1, so that

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + i^2y_1y_2 + iy_1x_2 + ix_1y_2\\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2).\end{aligned}$$

We call the set of all elements $x + iy$ the set of *complex numbers*, and we denote this set by C. We saw in *Unit 27* that there is a subset of C which is isomorphic to R (for addition and multiplication) under addition and multiplication. So, by a small abuse of language, we can regard R as a subset of C. (Alternatively, we can say that C contains a "copy" of R.)

In sections 29.1–4 of this text we concentrate on functions from C to C, often called *complex functions*, and, in particular, on how they can be represented pictorially. It is possible to define complex forms of the well-known elementary functions exp, sin, cos, tan, ln, etc., which reduce to their real forms when their domain is restricted to R, regarded as a subset of C. In section 29.5 we define and discuss the square roots and nth roots of a complex number. We discuss the nth root mapping and elaborate on our statement in section 27.5 of *Unit 27* that every polynomial of degree n has exactly n complex roots.

It is not the purpose of the Foundation Course to do "everything": our main aim is to show you that a subject exists by describing some of its basic concepts, and thereby to open the door to further study. In this unit we shall do no more than introduce the exponential function and discuss some very simple examples of complex functions and mappings.

29.1 THE EXPONENTIAL FUNCTION

29.1

29.1.0 Introduction

29.1.0

Introduction
* *

The definition of multiplication in C has a natural interpretation in terms of the geometric notions of scaling and rotation. The polar co-ordinate system is very useful in this context because multiplication of complex numbers in polar form is particularly easy. To illustrate this statement we shall revise a result, discussed in *Unit 27*, which we shall use again in this unit. Suppose that we take a particular complex number $z = x + iy$ corresponding to the polar co-ordinates* (r, θ), so that

$$z = x + iy = r\cos\theta + ir\sin\theta;$$

then we know that, for any positive integer n, z^n corresponds to the polar co-ordinates $(r^n, n\theta)$. In other words,

$$\begin{aligned} z^n &= r^n \cos n\theta + ir^n \sin n\theta \\ &= r^n(\cos n\theta + i\sin n\theta). \end{aligned}$$

But we know that

$$\begin{aligned} z^n &= (r\cos\theta + ir\sin\theta)^n \\ &= r^n(\cos\theta + i\sin\theta)^n \end{aligned}$$

and therefore

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

This result is a special case of De Moivre's Theorem, and we shall use it in developing a definition of the complex exponential function.

29.1.1 Extending the Domain

29.1.1

Main Text
* * *

You will remember that in *Unit 7, Sequences and Limits I* we defined the exponential function with domain R, and our object now is to extend the domain of the exponential function to C. You will also remember that $\exp x$ could alternatively be written as e^x (where $e \simeq 2.71828$).

Obviously the extended exponential function

$$z \longmapsto \exp z \qquad (z \in C)$$

must coincide with the original function

$$x \longmapsto \exp x \qquad (x \in R)$$

when the domain is restricted to R. But this in itself is not a sufficient guide to suggest a definition of $\exp z$, so we shall specify some of the properties of $\exp x$ which we would like $\exp z$ to have also.

Unfortunately, the most obvious approach is not possible. In *Unit 7* we defined the exponential function by

$$\exp : x \longmapsto \lim_{k \text{ large}} \left(1 + \frac{x}{k}\right)^k \qquad (x \in R \text{ and } k \in Z^+).$$

But we have not defined limits in the context of complex numbers (although we could), so we cannot define the complex exponential function to be

$$\exp : z \longmapsto \lim_{k \text{ large}} \left(1 + \frac{z}{k}\right)^k \qquad (z \in C \text{ and } k \in Z^+),$$

although, if we gave an appropriate meaning to the limit, this would be a satisfactory definition.

* We use black brackets for polar co-ordinates now, as is usual in the mathematical literature. It should be clear from the context which system of co-ordinates we are using.

In *Unit 12, Differentiation I*, we found a characteristic property of the exponential function:

$$(\exp)' = \exp.$$

But unfortunately we have not defined the derived function of a complex function (although we could).

In *Unit 14, Sequences and Limits II* we found a convergent series for the exponential function:

$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (x \in R).$$

But we have not defined convergence of an infinite complex series (although we could), so we can't just replace x by z and R by C to obtain a definition of the complex exponential function.

We have recounted this tale of woe because, had we the necessary ancillary ideas, each of these three possibilities for a definition could provide us with a more satisfactory approach than the one we actually adopt. But we can make up what we lack in expertise by bravado.

In *Unit 7* we discussed a very important property of the exponential function:

$$\exp(x_1 + x_2) = \exp x_1 \times \exp x_2 \qquad (x_1, x_2 \in R),$$

that is, the real exponential function is a morphism (isomorphism) of $(R, +)$ to $(R^+, \times)$. Let us agree to preserve this property for a start; that is, we require of the complex exponential function that

$$\exp(z_1 + z_2) = \exp z_1 \exp z_2 \qquad (z_1, z_2 \in C).$$

This means that the complex exponential function is a morphism (perhaps not an isomorphism) of $(C, +)$ to $(C_1, \otimes)$, where C_1 is some, as yet undetermined, subset of C and $\otimes$ is the symbol we introduced for complex multiplication in *Unit 27*.

There are some immediate consequences. Let

$$z_1 = x \quad \text{and} \quad z_2 = iy \qquad (x, y \in R);$$

then

$$\exp(x + iy) = \exp x \exp(iy),$$

so for any complex number z, we know that

$$\exp z = e^x \exp(iy).$$

We can see that $\exp z$ falls into two parts, one of which is the real number e^x. Our problem now is to find a suitable definition of $\exp(iy)$, so we investigate this part further.

We have assumed that $\exp z$ is a complex number, so we can write

$$\exp(iy) = f(y) + ig(y),$$

where f and g are real functions.

Now let n be any positive integer. Then

$$\exp(iny) = (\exp(iy))^n,$$

by repeated application of our morphism property. So that

$$f(ny) + ig(ny) = (f(y) + ig(y))^n.$$

Compare this with the special case of De Moivre's Theorem cited in section 29.1.0:

$$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n.$$

The suggestion is clear, but there is no "proof" for any conclusion. So we resort to bravado and define

$$\exp(iy) = \cos y + i \sin y \qquad (y \in R).$$

Definition 1

So we define the complex exponential function by

$$\exp : x + iy \longmapsto e^x(\cos y + i \sin y) \qquad (x + iy \in C).$$

Definition 2

The first thing we must do is to check that the complex exponential function does reduce to the real exponential function when the domain is restricted to R. Indeed it does, for then $y = 0$ and

$$\exp(x + i0) = e^x(\cos 0 + i \sin 0),$$

so that

$$\exp x = e^x.$$

It seems, therefore, that this function has some of the desirable properties which we would expect from an extension of the real exponential function. When we meet complex differential calculus we shall also be able to verify that

$$(z \longmapsto \exp z)' = (z \longmapsto \exp z),$$

(where we use $'$ to indicate the derived function).

In order to achieve consistency with the real exponential function, it is quite common to put

$$\exp z = e^z.$$

(Strictly speaking, this does not mean "e to the power z" and we shall have to wait for a future discussion of complex analysis before we can adequately deal with numbers raised to complex powers. We shall find then that this notation for the exponential function is consistent with the notation for powers.)

The equation

$$\exp(iy) = \cos y + i \sin y$$

or

$$e^{iy} = \cos y + i \sin y$$

is known as Euler's formula.

There is a mathematical oddity which is interesting. If we put $y = \pi$ in Euler's formula, then we have

$$e^{i\pi} = (\cos \pi + i \sin \pi) = -1,$$

so that

$$e^{i\pi} + 1 = 0.$$

This equation contains five of the most significant numbers in the history of mathematics in a neat and tidy formula: e, i, π, 1 and 0. We could devote a course unit to each of them.

It is important to notice that the complex exponential function embodies a neat relationship between a complex number $z = x + iy$ and the corresponding polar co-ordinates (r, θ). We know already that

$$x = r \cos \theta,$$

and

$$y = r \sin \theta,$$

so that

$$z = r(\cos \theta + i \sin \theta);$$

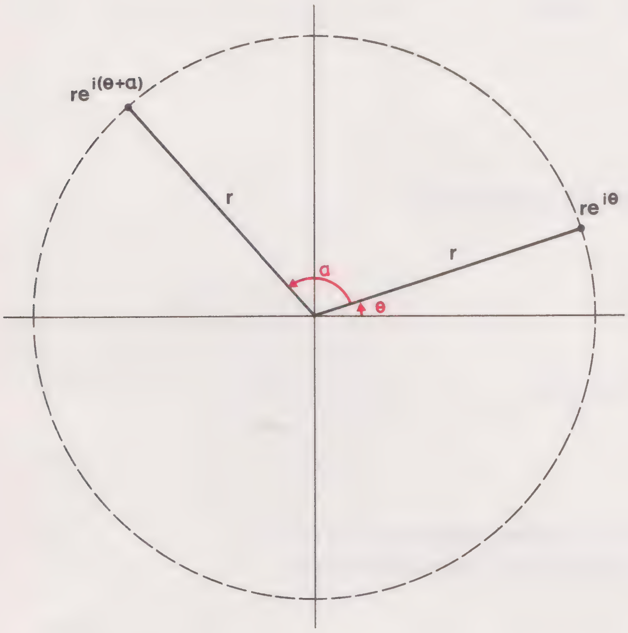
using Euler's formula we can now write this as

$$z = re^{i\theta}.$$

We shall use z as an abbreviation for each of the various ways of writing a complex number: (x, y), $x + iy$, (r, θ), $r(\cos\theta + i\sin\theta)$ and $re^{i\theta}$. This will not lead to any confusion for each representation refers to the same point z in the complex plane.

Notice particularly that multiplying a complex number by $e^{i\alpha}$ (for any real number α) has the effect of rotating the corresponding geometric vector anti-clockwise through an angle α about the origin; in other words, increasing the argument of the complex number by α, for

$$\begin{aligned} e^{i\alpha}z &= re^{i\alpha}e^{i\theta}, \\ &= re^{i(\alpha+\theta)}, \\ &= r(\cos(\alpha+\theta) + i\sin(\alpha+\theta)). \end{aligned}$$



Exercise 1

Exercise 1
(4 minutes)

(i) Show that if $z = e^{i\theta}$, then $\bar{z} = e^{-i\theta}$.
(ii) Show that $|e^{i\theta}| = 1$ for all real numbers θ.
(iii) Show that $|e^z| = e^x$.
(iv) Show that $e^{z+i2\pi} = e^z$ for all complex numbers z.
(v) Find the set of all complex numbers z for which $e^z = 1$.

(HINT: You may find it helpful to refer to the Summary at the end of section 27.4.2, *Unit 27*.) ■

Exercise 2

Exercise 2
(3 minutes)

Show that if

$$z = re^{i\theta} \qquad r \neq 0,$$

then

$$\frac{1}{z} = \frac{1}{r}e^{-i\theta}$$ ■

Exercise 3

Exercise 3
(3 minutes)

(i) Is the complex exponential function an isomorphism or a homomorphism from $(C, +)$ to $(C_1, \otimes)$?
(ii) What is the image set C_1?

(HINT: Some of the results of Exercise 1 will prove useful.) ■

Solution 1

Solution 1

(i) If

$$z = \cos\theta + i\sin\theta,$$

then

$$\begin{aligned}\bar{z} &= \cos\theta - i\sin\theta\\ &= \cos(-\theta) + i\sin(-\theta)\\ &= e^{i(-\theta)}.\end{aligned}$$

(ii) We use the property:

$$|z|^2 = z\bar{z}.$$

$$\begin{aligned}|e^{i\theta}|^2 &= (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta),\\ &= \cos^2\theta + \sin^2\theta = 1;\end{aligned}$$

hence

$$|e^{i\theta}| = 1,$$

since the modulus of a complex number is non-negative.

(iii)
$$\begin{aligned}|e^z| &= |e^{x+iy}|\\ &= |e^x e^{iy}|\\ &= |e^x||e^{iy}|.\end{aligned}$$

From (ii), $|e^{iy}| = 1$, and since $|e^x| = e^x$, we have

$$|e^z| = e^x.$$

(iv)
$$e^{z+i2\pi} = e^z e^{i2\pi}.$$

Multiplying by $e^{i2\pi}$ corresponds to a rotation about the origin through an angle of 2π radians anti-clockwise and this demonstrates the required result.

Notice also that we could argue that

$$e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1$$

so that

$$e^{z+i2\pi} = e^z e^{i2\pi} = e^z.$$

(v) If $e^z = 1$ and $z = x + iy$, then

$$e^x(\cos y + i\sin y) = 1.$$

Hence

$$e^x\cos y = 1 \qquad \text{(a)}$$

and

$$e^x\sin y = 0 \qquad \text{(b)}.$$

From (b) we have $y = n\pi$, $n \in Z$, and substituting in (a) we obtain

$$e^x = \pm 1.$$

But x is real and there is no real x such that $e^x = -1$, so $e^x = +1$. Hence

$$x = 0.$$

From (a), it follows that

$$\cos y = 1,$$

so n is even.

Finally,

$$z = i2k\pi, \qquad k \in Z,$$

and, just as we would expect, the solution set corresponds to the set of rotations about the origin through multiples of 2π anti-clockwise. ■

Solution 2

Solution 2

$$\begin{aligned}(re^{i\theta})\left(\frac{1}{r}e^{-i\theta}\right) &= e^{i\theta}e^{-i\theta} \\ &= (\cos\theta + i\sin\theta)(\cos(-\theta) + i\sin(-\theta)) \\ &= (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) \\ &= 1.\end{aligned}$$

If $z = re^{i\theta}$, then it follows that $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$. ■

Solution 3

Solution 3

(i) As a result of Exercise 1, part (iv), we can conclude that the complex exponential function is many-one. So it is a homomorphism, as opposed to the real exponential function which is an isomorphism.

(ii) By definition,

$$e^z = e^x(\cos y + i\sin y).$$

Therefore, e^z can be written as (e^x, y) in polar co-ordinates. (The modulus of e^z is e^x, see Exercise 1, part (iii), and hence y is one element of $\arg(e^z)$.)

Now e^x can be any positive number and y can be any real number, and we shall exhaust all the images by allowing y to take all values in the interval $[0, 2\pi[$. So (e^x, y) can be the polar co-ordinates of any point in the plane except $(0, 0)$. Hence the image set C_1 is C without the zero element. ■

29.2 REPRESENTATION OF COMPLEX FUNCTIONS

29.2

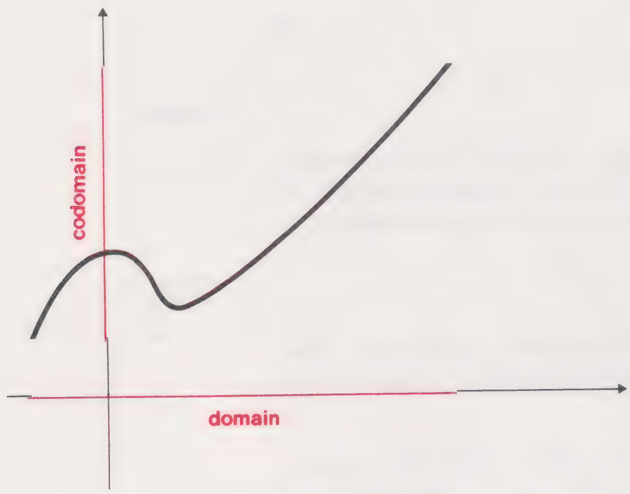
29.2.0 Introduction

29.2.0

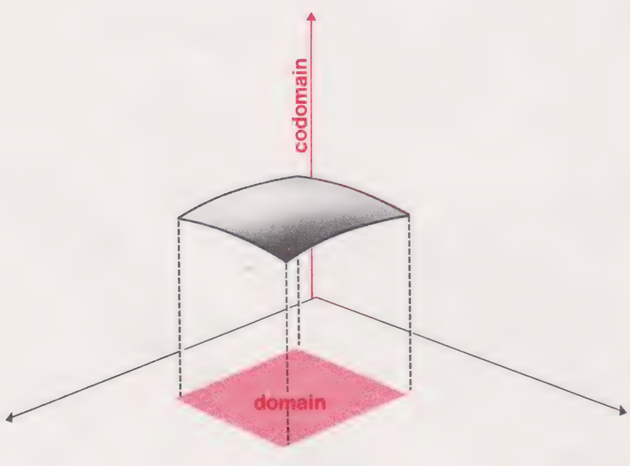
Introduction
* *

In the previous section we showed, in a particular case, how we can extend the definition of a real function to obtain a complex function. Having obtained our complex function, we need to examine some of its properties. This we did for the exponential function in the exercises, using purely algebraic methods. When looking at a particular real function in the past, one of the first things we did was to draw its graph. We shall now investigate what replaces the graph when we are dealing with complex functions.

We are very familiar with the fact that it is often possible to represent a function of one real variable by a graph:



We have also seen how it is often possible to represent a function of two real variables by a surface:

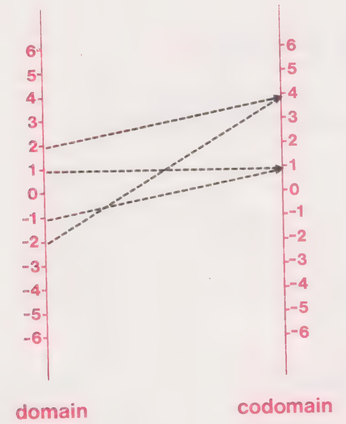


Representations of this kind are very useful because they give us an intuitive insight into the behaviour of the functions. We would like to produce something similar for complex functions, but there is one major problem. Suppose that we have a function $\phi: C \longrightarrow C$. We know that

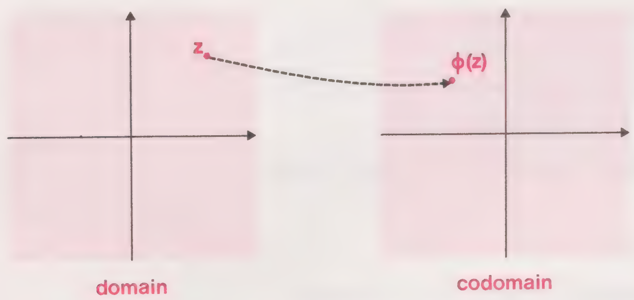
C can be represented by an Argand diagram, but to obtain a representation of ϕ similar to the graph of a real function, we need *two* copies of C, one for the domain of ϕ and the other for the codomain. A direct extension of the graph would only be feasible for the inhabitants of a four-dimensional universe.

The answer is to extend the "ladder" diagram which we have previously used for functions of one real variable, for example, the "square" function:

$$x \longmapsto x^2 \qquad (x \in R).$$



In this diagram the domain and codomain are kept separate and we indicate the corresponding points on the two sets. For complex functions we have *two* Argand diagrams, one for the domain and one for the codomain.



the function ϕ

Each point in the domain will be mapped to a corresponding point in the codomain by the complex function ϕ, and we can indicate the corresponding points on the two Argand diagrams.

You may find this way of representing complex functions a little difficult at first. But once you have drawn a few such diagrams and used them to analyse simple functions, you will begin to find this method of representation very useful. In the next section we consider a simple example: our friend the "square" function.

29.2.1

Main Text

29.2.1 The "Square" Function

As an example of the development of the representation of complex functions, we examine points and their images under the "square" function:

$$z \longmapsto z^2 \qquad (z \in C).$$

It is very often useful to let $z = x + iy$ denote the (complex) variable in the domain, and $w = u + iv$ denote the corresponding variable in the codomain, so that in this case we have

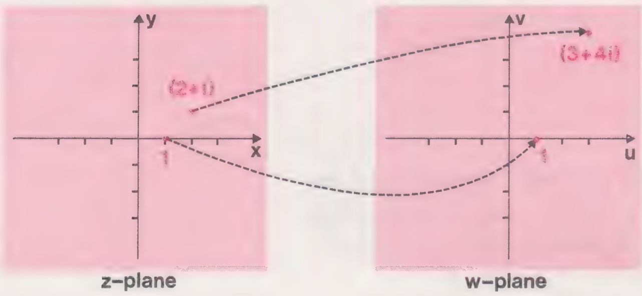
$$w = z^2 \qquad (z \in C).$$

The domain and codomain are then usually referred to as the z-plane and the w-plane respectively.

We get some intuitive feeling for the "square" function if we plot various points and their images; for example,

if $z = 1$ then $w = 1$,

if $z = 2 + i$ then $w = 3 + 4i$.

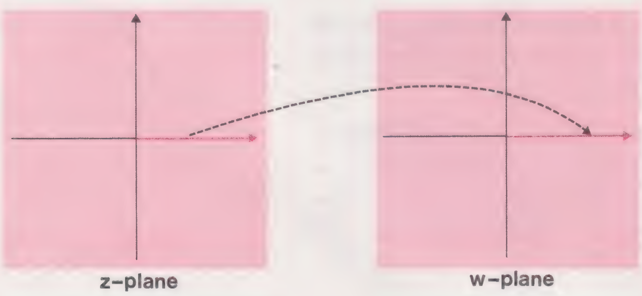


Isolated points are not sufficient to convey the behaviour of the function, but if we consider the images of a *set of points* then we can often see the properties of the function more clearly. For example, what is the image of the set of points which lie on the positive real axis in the z-plane?

The positive real axis in the z-plane is the set $\{(x, 0): x > 0\}$, and since $w = z^2$ we know that in the image set

$$w = u + iv = (x + i0)^2 = x^2.$$

Therefore the image set is the set of points for which u is positive and $v = 0$, i.e. $\{(u, 0): u > 0\}$, the positive real axis in the w-plane.



Now let us examine the image of a semi-circle in the upper half-plane with centre at the origin and radius 1. This set is most easily specified in terms of polar co-ordinates; it is the set of complex numbers whose

polar co-ordinates belong to the set

$$\{(r, \theta) : r = 1, 0 \leqslant \theta \leqslant \pi\}.$$

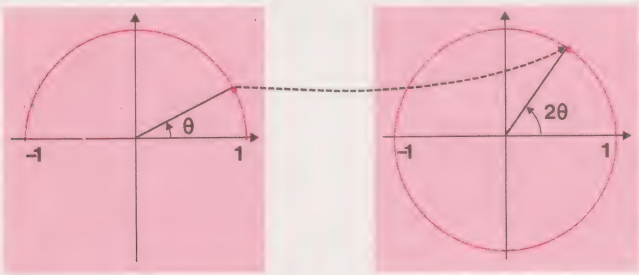
Since the "square" function is also very easily specified in terms of polar co-ordinates:

$$(r, \theta) \longmapsto (r^2, 2\theta),$$

it seems best to work entirely in this co-ordinate system. It follows that

$$\{(r, \theta) : r = 1, 0 \leqslant \theta \leqslant \pi\} \longmapsto \{(r, \theta) : r = 1^2, 0 \leqslant \theta \leqslant 2\pi\},$$

so the image of the semi-circle in the z-plane is a complete circle in the w-plane.

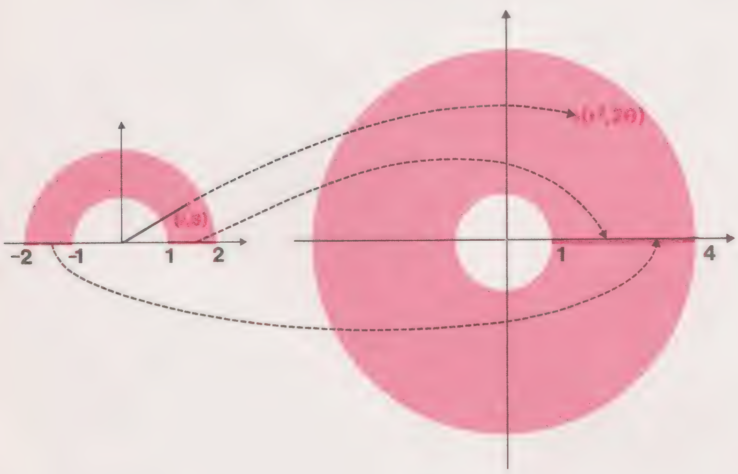


We can also obtain useful information from looking at the images of *regions* in the domain. For example, consider the set of complex numbers specified by the set

$$\{(r, \theta) : 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\},$$

which represents an annular region. Each point specified by (r, θ) in the domain is mapped to $(r^2, 2\theta)$ by the "square" function, so the image set is the set of complex numbers

$$\{(r, \theta) : 1 \leqslant r \leqslant 4, 0 \leqslant \theta \leqslant 2\pi\}.$$



This set of complex numbers is more neatly expressed as the set

$$\{z : 1 \leqslant |z| \leqslant 4\}.$$

Exercise 1

Exercise 1
(3 minutes)

Draw the image of the line segment

$$\{z : 1 \leqslant y \leqslant 2, x = 0\}$$

under the "square" function. ■

Solution 1

Solution 1

We have

$$z = iy \quad \text{where } 1 \leqslant y \leqslant 2,$$

so

$$w = z^2 = -y^2,$$

i.e.

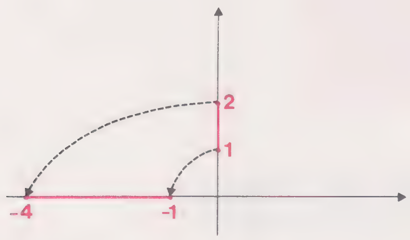
$$w = u + iv$$

where

$$u = -y^2 \quad \text{and} \quad v = 0.$$

So the image set is

$$\{(u, 0): -4 \leqslant u \leqslant -1\}.$$



In this diagram the domain and codomain are superimposed.

■

29.2.2 Representation of Complex Functions

29.2.2

Discussion
★ ★

You may find it difficult to see how we actually determine the image set under a particular function. Sometimes it is easy geometrically with little or no algebraic manipulation, as in the case of the "square" function, but often we are forced to use an algebraic approach, interpreting the algebra at the end to obtain the geometric picture.

Suppose that we have a function

$$F: z \longmapsto w \qquad (z \in C).$$

A set in the domain is determined by some restriction on the values of z; under the function F this is converted to a restriction on the values of w, and hence we determine our set in the codomain. Diagrammatically we have:

$$\begin{tikzcd}
\boxed{\text{a restriction on the values of } z} \arrow[r, "\text{determines}"] \arrow[d, "F"'] & \boxed{\text{a set in the domain of } F} \arrow[d, "F"] \\
\boxed{\text{a restriction on the values of } w} \arrow[r, "\text{determines}"] & \boxed{\text{the corresponding set in the codomain}}
\end{tikzcd}$$

For example, what is the image of the circle of unit radius, centred at the origin, under the function

$$z \longmapsto \frac{1+z}{1-z} \qquad (z \in C,\ z \neq 1)?$$

The circle is determined by the equation

$$|z| = 1 \qquad z \neq 1,$$

and this is our restriction on z. (Notice that we cannot have the complete circle, because we have not included $z = 1$ in the domain of our function.)

We put

$$w = \frac{1+z}{1-z}$$

so that

$$w - zw = 1 + z$$

and hence

$$z = \frac{w-1}{w+1} \qquad w \neq -1.$$

We can now substitute for z in the equation representing the restriction, to obtain

$$\left|\frac{w-1}{w+1}\right| = 1 \qquad w \neq -1,$$

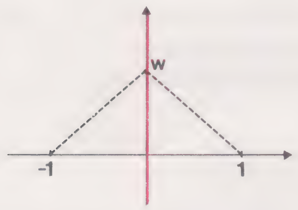
and this is a restriction on the values of w in the codomain. It only remains to give a geometric interpretation of this set. If we rearrange the equation, we get

$$|w-1| = |w+1| \quad w \neq -1.$$

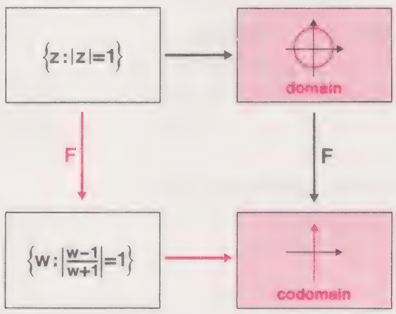
We know* that $|w - a|$ means the distance of w from a. We can now read the equation as

> "the distance of w from 1 equals the distance of w from -1, where $w \neq -1$".

The points equidistant from 1 and -1 clearly lie on the perpendicular bisector of the line joining 1 and -1, which is the imaginary axis. (The condition $w \neq -1$ does not exclude any of these points.)



Using the same diagrammatic representation as in the general discussion, we have



Notice that at one stage we need to find z in terms of w, which effectively means finding the reverse function. In our example the function is one-one, but in other cases it might be less straightforward; for example, this technique might not be effective for the "square" function which is a many-one function.

There is one point which we have glossed over in the discussion so far, which is best explained in terms of the example we have just considered.

We know that every point of the unit circle (except the point $z = 1$) in the z-plane maps to a point on the imaginary axis in the w-plane. We can express this by

$$|z| = 1, \qquad z \neq 1 \Rightarrow |w - 1| = |w + 1|, \qquad w \neq -1$$
$$\Rightarrow u = 0,$$

where $w = u + iv$.

But we have not checked that the image of the unit circle is the *whole* of the imaginary axis, that is,

$$u = 0 \Rightarrow |z| = 1 \quad \text{and} \quad z \neq 1.$$

Intuition would suggest that it is so, but intuition can be very deceptive when dealing with complex functions. Also, in this particular case there

* *Unit 27, Complex Numbers I*, Exercise 27.4.2.2.

is rather an awkward hole in the unit circle at $z = 1$. As it happens, it is not very difficult to check: we have

$$z = \frac{w-1}{w+1} \qquad w \neq -1;$$

putting $u = 0$, we get

$$z = \frac{iv-1}{iv+1}$$

whence, using the fact that $|w_1| \times |w_2| = |w_1 w_2|$ with $w_1 = iv - 1$ and $w_2 = \dfrac{1}{iv+1}$, we have

$$|z| = \frac{\sqrt{v^2+1}}{\sqrt{v^2+1}} = 1.$$

So *every* point for which $u = 0$ arises from *some* point for which $|z| = 1$.

In the television programme associated with this unit, we illustrate the last statement by using computer animation. As a point traverses the given set in the domain, we see the corresponding point traversing the image set in the codomain. This is a particular advantage of the method of displaying the function by computer animation. Another point which is worth noticing in these computer animations is that while the point in the domain moves with uniform speed, the image point in the codomain will often speed up or slow down. This illustrates very clearly the way in which subsets of the domain correspond to subsets of the image set in the codomain. All this would be very difficult to describe in the correspondence text, but it does give a further bonus to the amount of visual information we can get from this method of representation. If you have watched the television programme before reading this text, we advise you to watch it again (if possible), paying special attention to the computer animations.

Exercise 1

Exercise 1
(4 minutes)

Find the image of each of the following sets under the "square" function

$$z \longmapsto z^2 \qquad (z \in C).$$

(i) $\{z : x < 0, y \in R\}$
(ii) $\{z : x = 1, y \in R\}$
(iii) $\{z : x \in R, y = 1\}$ ■

Exercise 2

Exercise 2
(3 minutes)

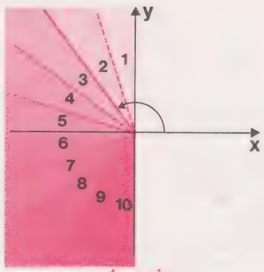
Find the image of the circle centred at the origin with unit radius under the function

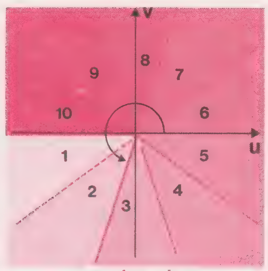
$$z \longmapsto 2z + 3 \qquad (z \in C).$$ ■

Solution 1

Solution 1

(i)





The image set is the complex plane with the negative real axis (including the origin) removed. We can easily see that this is so if we notice that the image of every radial line through the origin is also a radial line through the origin but with twice the argument.

(ii) If $z = x + iy$ and $w = u + iv$, then since $w = z^2$, we have

$$u = x^2 - y^2,$$

$$v = 2xy.$$

If $x = 1$, then

$$u = 1 - y^2$$

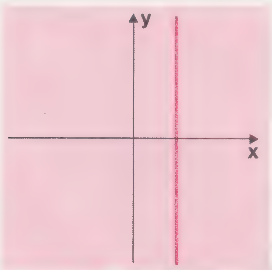
$$v = 2y.$$

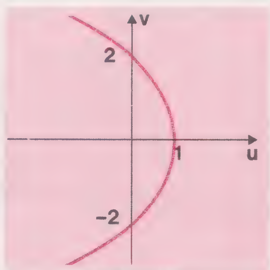
To obtain the image set in the w-plane, we eliminate y between these two equations, to obtain

$$1 - u = \frac{v^2}{4},$$

which you may recognize as the equation of a parabola. (If you do not recognize the equation, then we suggest you sketch the graph; you can easily check that the complete parabola corresponds to the line.)

(See RB4)





(iii) If $y = 1$, then

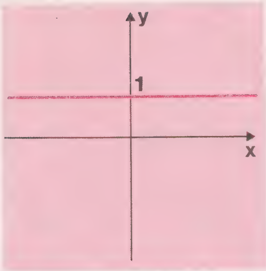
$$u = x^2 - 1,$$

$$v = 2x,$$

so that

$$u = \frac{v^2}{4} - 1$$

which is again the equation of a parabola.



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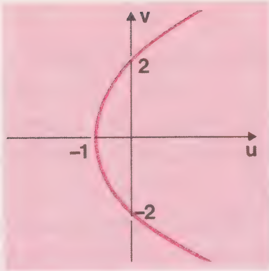


codomain

■

Solution 2

Solution 2

Let $w = 2z + 3$. The given circle is the set $\{z : |z| = 1\}$.

We have

$$z = \frac{w - 3}{2},$$

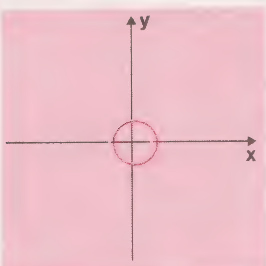
and therefore the image set is determined by the restriction

$$\left|\frac{w - 3}{2}\right| = 1,$$

which simplifies to

$$|w - 3| = 2.$$

This means that "the distance of w from 3 is 2", so the image set is a circle centred at 3 with radius 2.



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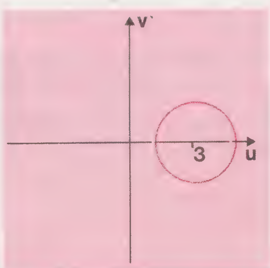


codomain

■

29.2.3 Invariance

29.2.3

Main Text

We first mentioned the concept of invariance in *Unit 23, Linear Algebra II*, section 23.1.1. Invariance is another of the notions which we introduce in this course because it occurs widely in mathematics. Often in mathematics it is very interesting and useful to ask what is left undisturbed, or invariant, under a function. For example, the real exponential function $x \longmapsto e^x$ $(x \in R)$ is invariant under the differentiation operator D. We have in fact already seen many different examples of invariance. We have devoted part of the radio programme which forms part of this unit to a discussion of invariance.

In the context of complex functions, invariant points and sets of points can be of considerable assistance in the visualization of the functions. An invariant point of the complex function f is a point a such that

$$f(a) = a.$$

An invariant set under the function f is a set A such that

$$f(A) = A.$$

Notice that in the latter case the individual points of A need not themselves be invariant; that is, we do not require that

$$f(a) = a \quad \text{for all } a \in A,$$

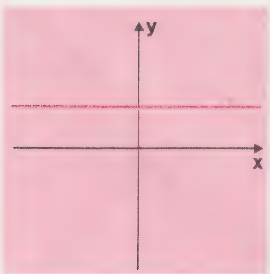
but we do require that

$$\{f(a): a \in A\} = A.$$

Notice also that, although we use two Argand diagrams to represent a complex function, the idea of invariance is expressed more naturally in terms of one; i.e. we are regarding the codomain as superimposed on the domain. For example, consider the function

$$z \longmapsto z + 1 \qquad (z \in C),$$

which translates every point z one unit parallel to the real axis. We know that this function has no invariant points, but it does have invariant sets of points. For example, any line parallel to the real axis is invariant.



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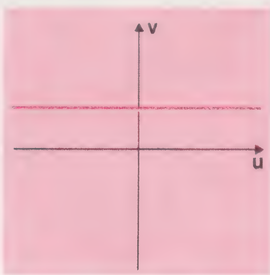


codomain

Although each point moves to a new point, the set of all points lying on such a line remains unchanged.

Exercise 1

Exercise 1
(3 minutes)

Which *points* are invariant under the following functions?

(i) $z \longmapsto z + z_0 \qquad z_0 \in C \qquad (z \in C)$

(translation function)

(ii) $z \longmapsto e^{i\alpha}z \qquad \alpha \text{ real} \qquad (z \in C)$

(rotation function)

(iii) $z \longmapsto kz \qquad k \text{ real} \qquad (z \in C)$

(scaling function) ■

Exercise 2

Exercise 2
(4 minutes)

Find examples of *sets* which are invariant under the functions

(i) $z \longmapsto e^{i\alpha}z \qquad \alpha \text{ real} \qquad (z \in C)$

(ii) $z \longmapsto kz \qquad k \text{ real} \qquad (z \in C)$ ■

Main Text

There are two sets of a different sort which are invariant under all three functions:

$$z \longmapsto z + z_0,$$

$$z \longmapsto e^{i\alpha}z,$$

$$z \longmapsto kz,$$

and which are of particular interest, namely

the set of all circles

and

the set of all straight lines.

It isn't difficult to see that the image of a circle must be a circle under each of these functions, and the same is obviously true of straight lines. It is equally clear that the union of these sets, that is,

the set of all circles and straight lines,

is also invariant.

We are particularly interested in the last set because it is also invariant under the following function which we shall examine in the next section:

$$z \longmapsto \frac{1}{z} \qquad (z \in C, z \neq 0).$$

Solution 1

Solution 1

We consider these functions using their geometrical interpretation. They could just as easily be tackled algebraically. For instance, if in (i) we put

$$z = z + z_0,$$

we conclude that *either* $z_0 = 0$, in which case every point is invariant, *or* there is no invariant point.

(i) There are no invariant points unless $z_0 = 0$, in which case every point of C is invariant. Every point z is translated through a distance $|z_0|$ in a direction determined by $\operatorname{Arg}(z_0)$.
(ii) If α is a multiple of 2π, then every point of C is invariant: if not, then the only invariant point is the origin.
(iii) If $k = 1$, then every point of C is invariant: if not, then the only invariant point is the origin. ■

Solution 2

Solution 2

Any circle with centre at the origin is invariant under $z \longmapsto e^{i\alpha}z$, and any straight line through the origin is invariant under $z \longmapsto kz$. ■

29.3 THE BILINEAR FUNCTION

29.3

29.3.0 Introduction

29.3.0

Introduction

In this section we shall consider some further ideas in the geometric representation of complex functions, by considering in some detail the so-called bilinear functions, i.e. functions of the form

Definition 1

$$z \longmapsto \frac{az+b}{cz+d} \qquad \left(z \in C, z \neq -\frac{d}{c}\right),$$

where a, b, c and d are complex numbers such that $ad - bc \neq 0$. The reason for considering these particular functions is far from obvious, but it turns out that they represent a very important class of "well-behaved" complex functions. As far as we are concerned here, they are a convenient vehicle for continuing our discussion. If you are interested in their wider importance, and have the time, you can consult R. V. Churchill, *Complex Numbers and Applications* (McGraw-Hill 1960).

We begin by considering an important special case in which $a = d = 0$ and $b = c$:

$$z \longmapsto \frac{1}{z} \qquad (z \in C, z \neq 0).$$

29.3.1 The Function $z \longmapsto \frac{1}{z}$

29.3.1

Main Text

We shall denote the function

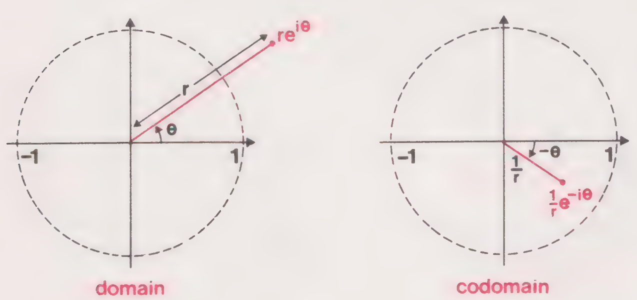
$$z \longmapsto \frac{1}{z} \qquad (z \in C, z \neq 0)$$

by Φ.

If we use the form $z = re^{i\theta}$, for convenience, then

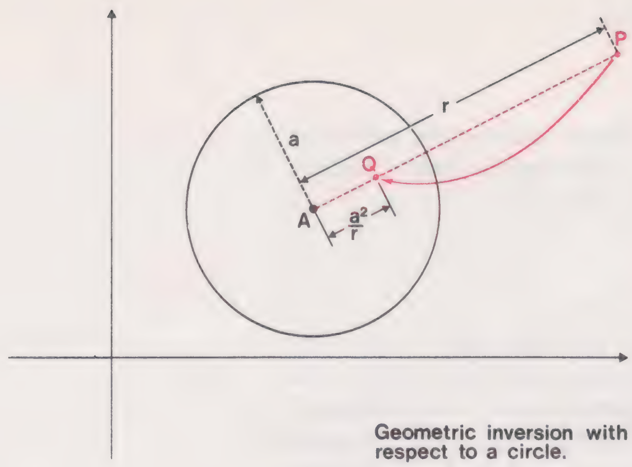
$$\Phi : re^{i\theta} \longmapsto \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.$$

(See Exercise 29.1.1.2.)



Φ can be interpreted in terms of a well-known geometric transformation. The function illustrated in the following diagram, in which, for example, $P \longmapsto Q$, is often called a geometric inversion with respect to the circle, centre A.

Definition 1



Geometric inversion with respect to a circle.

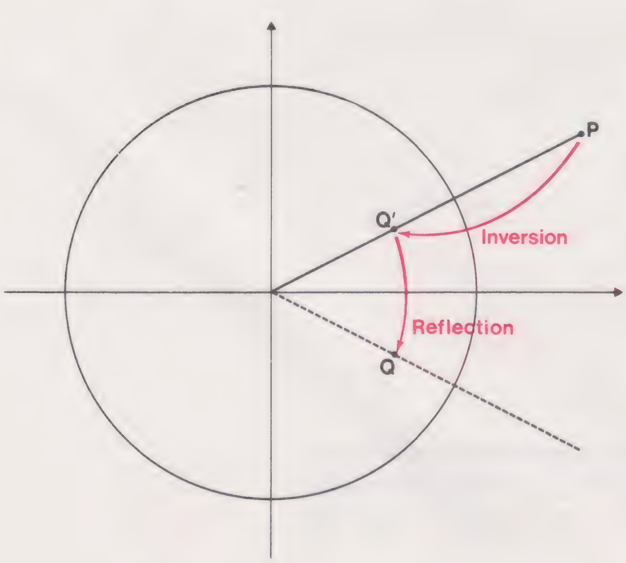
P and Q are known as a pair of inverse points with respect to the circle. These points have the property that A, P and Q are collinear and $AQ \cdot AP = a^2$, where a is the radius of the circle. It is clear that the function is one-one, and that points inside the circle map to points outside the circle and vice versa, except for the centre of the circle which has no image since it is not included in the domain of the function. Points on the circle map to themselves, that is, they are invariant points of the transformation.

Definition 2
* *

If we now look at our function Φ, then we see that it is very similar to a geometric inversion. If P represents z, Q represents $\frac{1}{z}$, A is the origin O and $a = 1$, then

$$OP \times OQ = |z| \times \left|\frac{1}{z}\right| = 1,$$

which suggests inversion in the unit circle centred at the origin. The only trouble is that O, P and Q are not, in general, collinear, since $\arg\left(\frac{1}{z}\right) = -\arg z$. If P and Q' are inverse points with respect to the circle $|z| = 1$, then Q is the reflection of Q' in the real axis.



Under the function $\Phi : z \longmapsto \frac{1}{z}$ we have, for example, $P \longmapsto Q$ in the last diagram. So we see that

> the function $z \longmapsto \frac{1}{z}$ can be interpreted as a geometric inversion with respect to the circle centred at the origin, with unit radius, followed by a reflection in the x-axis.

Exercise 1

Exercise 1
(3 minutes)

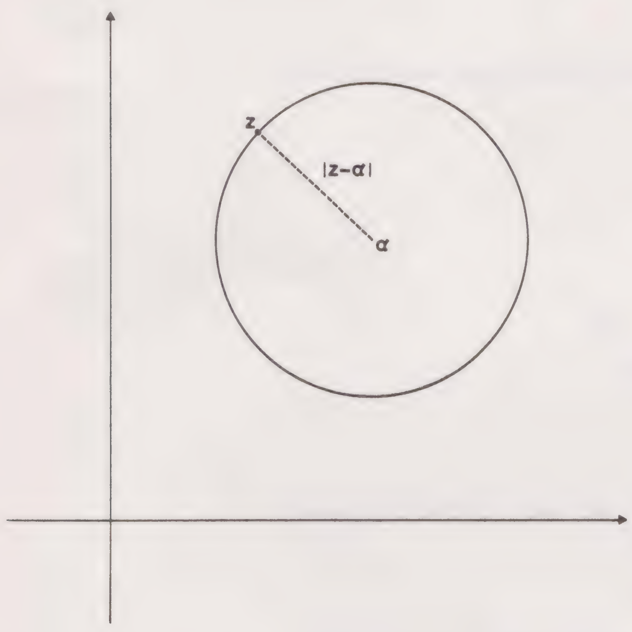
Which of the following sets are invariant under the function Φ?

(i) a straight line through the origin;
(ii) a straight line parallel to the imaginary axis;
(iii) a straight line parallel to the real axis;
(iv) the circle $\{z : |z| = a\}$, where $a \in R^+$;
(v) $\{e^{i\theta}, e^{-i\theta}\}$;
(vi) $\left\{z : \frac{1}{a} \leqslant |z| \leqslant a\right\}$, where $a \in R$ and $a \geqslant 1$. ■

Main Text

We now investigate algebraically what happens to circles and lines under the function Φ. Let's start with circles: a circle with radius a, and centre at the point representing the complex number α, has the equation

$$|z - \alpha| = a.$$



We can rewrite this equation as

$$(z - \alpha)(\bar{z} - \bar{\alpha}) = a^2$$

or

$$z\bar{z} - \bar{\alpha}z - \alpha\bar{z} = a^2 - \alpha\bar{\alpha}.$$

Under the function $\Phi : z \longmapsto \frac{1}{z}$ this becomes

$$\frac{1}{z\bar{z}} - \frac{\bar{\alpha}}{z} - \frac{\alpha}{\bar{z}} = a^2 - \alpha\bar{\alpha}$$

i.e.

$$1 - \bar{\alpha}\bar{z} - \alpha z = (a^2 - \alpha\bar{\alpha})z\bar{z}.$$

(*continued on page 24*)

Solution 1

Solution 1

(i) Since the origin is the centre of inversion, any point on a straight line through the origin maps under a geometric inversion to another point on the same straight line. The reflection of this line in the real axis is a straight line through the origin; the reflection of the line coincides with itself if and only if the line is either the real or the imaginary axis. Only the two axes are invariant.

(ii) A straight line parallel to the imaginary axis is not invariant, unless that straight line is the imaginary axis itself.

(iii) A straight line parallel to the real axis is not invariant, unless that straight line is the real axis itself.

(iv) The image of the set $\{z : |z| = a\}$ is the set $\left\{z : |z| = \dfrac{1}{a}\right\}$. So the only invariant set is the unit circle.

(v) $e^{i\theta}$ and $e^{-i\theta}$ are images of each other, so the set $\{e^{i\theta}, e^{-i\theta}\}$ is invariant.

(vi) This set is an annular region centred at the origin; it is invariant. ■

(*continued from page 23*)

Now if $a^2 - \alpha\bar{\alpha} \neq 0$, we can rearrange this as

$$z\bar{z} + \frac{\alpha}{a^2 - \alpha\bar{\alpha}}z + \frac{\bar{\alpha}}{a^2 - \alpha\bar{\alpha}}\bar{z} = \frac{1}{a^2 - \alpha\bar{\alpha}}.$$

If we compare this with the (red) equation of the original circle, we see that it is the same, except that

$$\alpha \text{ has been replaced by } \frac{-\bar{\alpha}}{a^2 - \alpha\bar{\alpha}}.$$

and

$$a^2 - \alpha\bar{\alpha} \text{ has been replaced by } \frac{1}{a^2 - \alpha\bar{\alpha}}.$$

So the image of a circle is a circle, in general.

There is, however, one exceptional case:

$$a^2 - \alpha\bar{\alpha} = 0.$$

What is special about this case? Well, $\alpha\bar{\alpha} = |\alpha|^2$ and $|\alpha|$ is the distance of the point representing α from the origin. So

$$a^2 - \alpha\bar{\alpha} = 0 \Rightarrow a = |\alpha|.$$

i.e. the circle $|z - \alpha| = a$ passes through the origin. (And if we had wanted to predict this special case we could have done so, since the origin does not belong to the domain of Φ. In this case there is, so to speak, a *hole* in our original circle.)

When $a^2 - \alpha\bar{\alpha} = 0$, the transformed equation becomes

$$1 - \bar{\alpha}\bar{z} - \alpha z = 0.$$

In the next exercise we ask you to interpret this equation.

Exercise 2

Exercise 2
(3 minutes)

By writing $z = x + iy$ and $\alpha = a + ib$, determine the set represented by the equation

$$1 - \bar{\alpha}\bar{z} - \alpha z = 0.$$ ■

Main Text
* * *

So we see that the image of a circle under $\Phi : z \longmapsto \frac{1}{z}$ is either a circle or a straight line. We could now go through the same investigation to find the image of a straight line. But with a little cunning there is no need to do this. The function Φ is not only one-one, but it is its own inverse. That is, if we apply Φ twice, then we end up where we started:

$$\Phi(\Phi(z)) = z.$$

We have seen that

Φ(*a circle through the origin*) = *a straight line,*

so that

a circle through the origin = Φ(*a straight line*).

We now wish to know *which* straight lines are the images of circles through the origin. This has been covered in Exercise 2; we showed that the circle through the origin, with centre $\alpha = a + ib$, is mapped to the line with equation

$$2by - 2ax + 1 = 0.$$

By varying the circle we can vary a and b and so obtain "almost any" straight line: the only type of line we cannot get is a line through the origin, because we can never get an equation of the form

$$cy + dx = 0.$$

There is no way of getting rid of the 1! So we know that

Φ(*any straight line not through the origin*) = *circle through origin.*

But we have already shown (Exercise 1, part (i)) that

Φ(*any straight line through origin*)

= *some straight line through origin.*

(Notice once again the crucial role of the origin.) This now covers all possible images of straight lines and circles, and we have seen that the image is always a straight line or a circle. That is, the set of all circles *and* straight lines is invariant under Φ. This completes our investigation of Φ.

We now turn to the composition of complex functions with the intention of expressing the general bilinear function as the composition of simpler and familiar functions.

Solution 2

Solution 2

$$1 - \bar{\alpha}\bar{z} - \alpha z = 1 - (a - ib)(x - iy) - (a + ib)(x + iy)$$
$$= 1 - 2ax + 2by$$

so that

$$1 - \bar{\alpha}\bar{z} - \alpha z = 0$$

becomes

$$2by - 2ax + 1 = 0,$$

which is the equation of a straight line. ■

29.3.2 Composition of Complex Functions

29.3.2

Main Text

We can combine complex functions just as we can combine real functions. For example, if

$$f: z \longmapsto z + 2 \qquad (z \in C)$$

and

$$g: z \longmapsto 3z \qquad (z \in C)$$

then

$$f + g: z \longmapsto 4z + 2 \qquad (z \in C).$$

Also

$$f \circ g: z \longmapsto 3z + 2 \qquad (z \in C)$$

and

$$g \circ f: z \longmapsto 3(z + 2) \qquad (z \in C).$$

Often it is very helpful, when considering an apparently complicated function, to break it down into the composite of two or more simple functions.

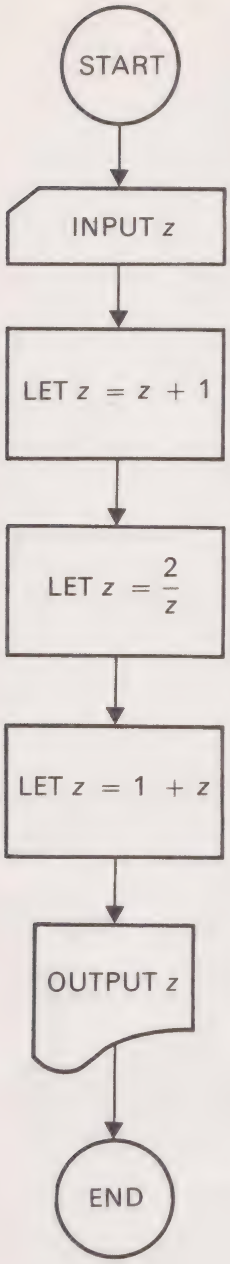
Let us look at the example

$$F: z \longmapsto \frac{z + 3}{z + 1} \qquad (z \in C, z \neq -1).$$

The first step could be to simplify the function to

$$F: z \longmapsto 1 + \frac{2}{z + 1} \qquad (z \in C, z \neq -1).$$

Now let us think of the simple processes which would enable us to calculate $F(z)$ for some arbitrary complex number $z \neq -1$. A "flow chart" representation of the calculation process is given on the next page.



Mapping Notation

$$z \longmapsto z + 1$$

$$z \longmapsto \frac{2}{z}$$

$$z \longmapsto 1 + z$$

Effectively this process is equivalent to the composition of the functions

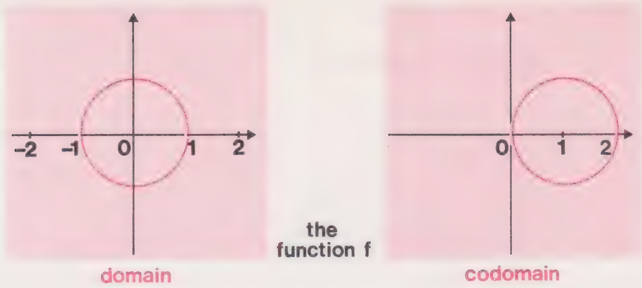
$$f : z \longmapsto z + 1 \qquad (z \in C),$$

$$g : z \longmapsto \frac{2}{z} \qquad (z \in C, z \neq 0).$$

We see that

$$F = f \circ g \circ f.$$

Suppose now that we take a particular set in the domain of F and attempt to find its image. For example, let us find the image of the circle $\{z : |z| = 1\}$. We must of course omit the point $(-1, 0)$ from the set since this point does not belong to the domain of F. (This point has not been omitted from the circle in the diagram as we cannot illustrate a "hole" at a single point.)



First we carry out the translation f, which simply moves the circle one unit to the right. Now take the new circle which is in the domain of g. (Notice that this time it is the point (0, 0) which is omitted from the circle.) The mapping g can itself be regarded as the composite $g_2 \circ g_1$ of

$$g_1 : z \longmapsto \frac{1}{z} \quad \text{and} \quad g_2 : z \longmapsto 2z.$$

We know what happens under $z \longmapsto \frac{1}{z}$ from our studies in the previous section. The circle passes through the origin, so it is mapped to the straight line with equation

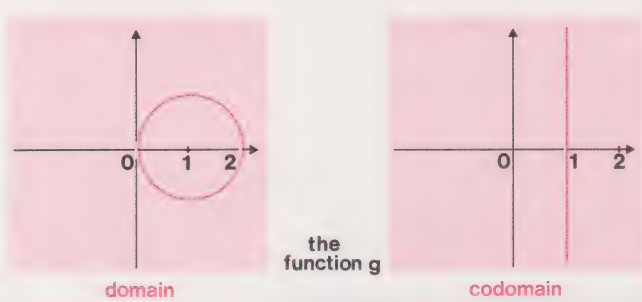
$$2by - 2ax + 1 = 0,$$

where (a, b) are the Cartesian co-ordinates of the centre of the circle. In this case the centre is $(1, 0)$, so the image line has equation

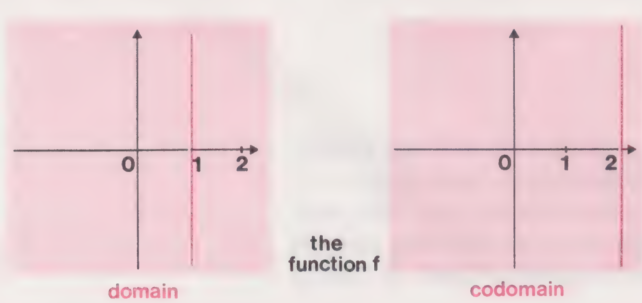
$$-2x + 1 = 0.$$

This is the line parallel to the imaginary axis passing through $(\frac{1}{2}, 0)$. This line is then mapped by the scaling $z \longmapsto 2z$, which doubles the distance of any point from the origin. So the image of the line is the line with equation

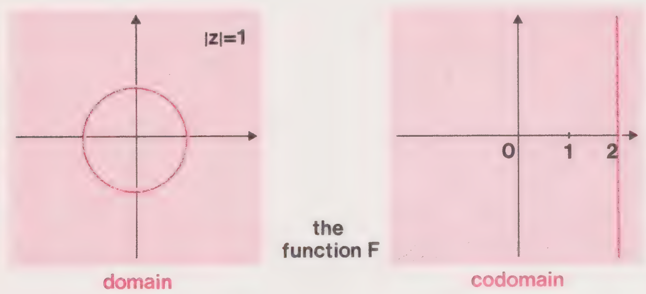
$$x = 1.$$



The next step is to find the image of this straight line under the function f. Since $f : z \longmapsto 1 + z$, we simply add 1 to each element in the domain, and therefore the line is moved one unit to the right.



We can now see that the original function $F = f \circ g \circ f$ maps the circle, with centre the origin and unit radius, to the line parallel to the y-axis through the point $(2, 0)$.



Exercise 1

Exercise 1
(3 minutes)

Find the image of the circle $\{z : |z| = 1\}$ under the following functions:

(i) $F : z \longmapsto \dfrac{3z}{z-1}$ $\quad (z \in C, z \neq 1)$

(ii) $G : z \longmapsto \left(2 + \dfrac{3z}{z-1}\right)$ $\quad (z \in C, z \neq 1)$

(iii) $H : z \longmapsto \left(\dfrac{3z}{z-1} - \dfrac{1}{2}\right)^2$ $\quad (z \in C, z \neq 1)$ ■

Solution 1

Solution 1

(i) Let

$$f:z \longmapsto z - 1,$$

$$g:z \longmapsto \frac{1}{z},$$

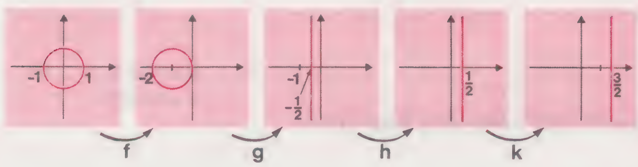
$$h:z \longmapsto 1 + z,$$

$$k:z \longmapsto 3z,$$

then

$$F = k \circ h \circ g \circ f.$$

Diagrammatically we have:



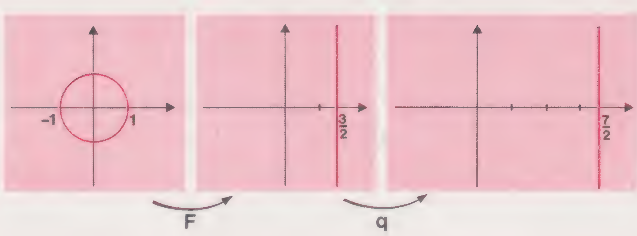
(ii) If

$$q:z \longmapsto 2 + z,$$

then

$$G = q \circ F.$$

Diagrammatically we have:



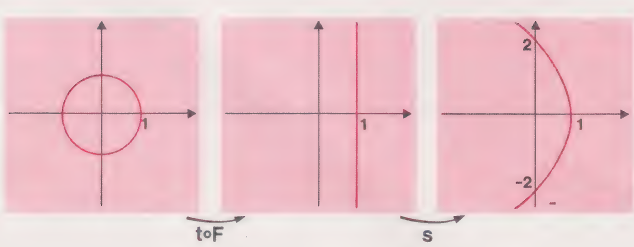
(iii) If

$$s:z \longmapsto z^2, \qquad t:z \longmapsto z - \tfrac{1}{2},$$

then

$$H = s \circ t \circ F.$$

Diagrammatically we have:



(For the last function see Exercise 29.2.2.1, part (ii).) ∎

29.3.3

Main Text

29.3.3 The Bilinear Function

The function

$$z \longmapsto \frac{az + b}{cz + d} \qquad \left(z \in C, z \neq -\frac{d}{c}\right),$$

where a, b, c, d are complex numbers and $ad - bc \neq 0$, is called a bilinear function. You may well ask why we insist that $ad - bc \neq 0$. The answer is simple: if $ad - bc = 0$, then $\frac{az + b}{cz + d}$ is either a constant or is meaningless.

If $ad - bc = 0$ and $bd \neq 0$, then

$$\frac{a}{b} = \frac{c}{d},$$

which means that

$$\frac{az + b}{b} = \frac{cz + d}{d}$$

so that

$$\frac{az + b}{cz + d} = \frac{b}{d}.$$

In other words, *every* point z maps to the single point $\frac{b}{d}$.

If $ad - bc = 0$ and $bd = 0$, then:

$$b = d = 0 \Rightarrow z \longmapsto \frac{a}{c};$$

$b = 0, d \neq 0 \Rightarrow z \longmapsto 0 \qquad (\text{since } a = 0);$

$b \neq 0, d = 0 \Rightarrow$ the function is not defined $\qquad (\text{since } c = 0).$

Clearly we wish to ensure at the outset that we do not have such catastrophic many-one functions.

The important property of a bilinear function is that it is the most general function which can be obtained as a composition of the elementary functions:

(i) $z \longmapsto z + z_0 \qquad\qquad (z \in C)$

(ii) $z \longmapsto e^{i\alpha} z \qquad \alpha$ real $\qquad (z \in C)$

(iii) $z \longmapsto kz \qquad k$ real $\qquad (z \in C)$

(iv) $z \longmapsto \frac{1}{z} \qquad\qquad (z \in C, z \neq 0).$

This isn't difficult to show, for if $c = 0$, then the function reduces to

$$z \longmapsto \frac{a}{d}z + \frac{b}{d} \qquad (z \in C),$$

and this is clearly a composition of the functions (i), (ii) and (iii). (Notice that d cannot also be zero, for then $ad - bc = 0$.)

On the other hand, if $c \neq 0$ then we can rearrange the terms in the expression for the image under the function to give

$$z \longmapsto \frac{a}{c} - \left(\frac{ad - bc}{c}\right)\left(\frac{1}{cz + d}\right).$$

We see that the bilinear function is the composite of the functions

$$z \longmapsto cz$$

$$z \longmapsto z + d$$

$$z \longmapsto \frac{1}{z}$$

$$z \longmapsto \left(\frac{ad - bc}{c}\right) z$$

$$z \longmapsto \frac{a}{c} - z$$

(in the above order, of course).

We can also show that if we take a composite of some of the functions

(i) $z \longmapsto z + z_0$

(ii) $z \longmapsto e^{i\alpha} z$

(iii) $z \longmapsto kz$

(iv) $z \longmapsto \frac{1}{z}$

then we always get a bilinear function (provided that we do not use (iii) with $k = 0$).

We need to ensure that if $z \longmapsto \dfrac{az + b}{cz + d}$ is the composite function, then $ad - bc \neq 0$. We leave you to verify this for yourself.

In general, if a set is invariant under a function f, and if that same set is invariant under a function g then it will also be invariant under $f \circ g$ and $g \circ f$.

We have shown that the set of all circles and straight lines is invariant under each of the functions

$$z \longmapsto z + z_0, \qquad z \longmapsto e^{i\alpha} z, \qquad z \longmapsto kz \quad \text{and} \quad z \longmapsto \frac{1}{z};$$

and, since a bilinear function is the composite of some of these functions, we know that

the set of all circles and straight lines is invariant under a bilinear function.

Exercise 1

Exercise 1
(4 minutes)

Find the inverse function of each of the four elementary functions. (You should first check that each function is one-one, i.e. that it has an inverse.) ■

Exercise 2

Exercise 2
(5 minutes)

(i) If T_1 and T_2 are bilinear functions, prove that $T_1 \circ T_2$ is a bilinear function.
(ii) If T is a bilinear function, show that the inverse function is also a bilinear function. ■

There are many other interesting properties of bilinear mappings which we do not have time to study in the Foundation Course. If you are interested in reading further, see R. V. Churchill, *Complex Variables and Applications* (McGraw-Hill 1960).

29.4 SOME SPECIAL FUNCTIONS

29.4

29.4.0 Introduction

29.4.0

Introduction

* *

In this section we intend to examine the behaviour of one or two particular functions. It is very useful for an applied mathematician, for example, to have a stock of information about particular functions, because his ability to find a solution to a particular problem may well depend on his ability to find a function which reduces a complicated shape in the domain to a simple shape (usually a circle) in the codomain. His strategy is essentially to map from a complicated situation to a simple one.

There is a fundamental *existence theorem*, due to Riemann, which, very roughly speaking, states that we can always map a region with a suitable boundary to a disc (the region bounded by a circle), using a "well-behaved" complex function. Unfortunately, although this theorem tells us that such functions *exist*, it does not tell us how to construct them, and for most practical purposes we are forced to rely on accumulated experience.

In section 29.4 we consider three special functions. If you are short of time, then work through section 29.4.3 only and just read quickly through sections 29.4.1 and 29.4.2. The latter section is also discussed, in part, in the television programme associated with this unit.

Bernhard Riemann
1826–1866
(Mansell Collection)

29.4.1 The Exponential Function

29.4.1

Discussion

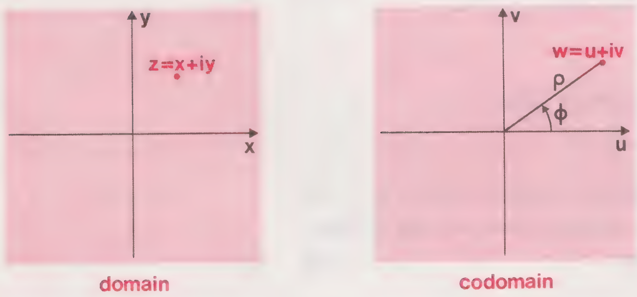
* *

We defined the complex exponential function

$$z \longmapsto \exp z$$

in section 29.1. If we let w be the image of z and (ρ, ϕ) be polar co-ordinates of w, then

$$w = \rho(\cos \phi + i \sin \phi)$$
$$= \rho e^{i\phi}$$



But

$$w = e^z = e^x e^{iy}$$

so that

$$\rho = e^x$$

and

$$e^{i\phi} = e^{iy}$$

(continued on page 35)

Solution 29.3.3.1

Solution 29.3.3.1

The inverse of

$$z \longmapsto z + z_0 \qquad (z \in C)$$

is

$$z \longmapsto z - z_0 \qquad (z \in C).$$

The inverse of

$$z \longmapsto e^{i\alpha}z \qquad \alpha \text{ real} \qquad (z \in C)$$

is

$$z \longmapsto e^{i(-\alpha)}z \qquad -\alpha \text{ real} \qquad (z \in C).$$

The inverse of

$$z \longmapsto kz \qquad k \text{ real}, k \neq 0 \qquad (z \in C)$$

is

$$z \longmapsto \frac{1}{k}z \qquad k \text{ real}, k \neq 0 \qquad (z \in C).$$

The function

$$z \longmapsto \frac{1}{z} \qquad (z \in C, z \neq 0)$$

is its own inverse.

We see that the inverse of each elementary function is an elementary function of the same form. ■

Solution 29.3.3.2

Solution 29.3.3.2

(i) T_1 and T_2 are composites of the elementary functions

$$z \longmapsto z + z_0, \qquad z \longmapsto e^{i\alpha}z, \qquad z \longmapsto kz \quad \text{and} \quad z \longmapsto \frac{1}{z},$$

so this must also be true of $T_1 \circ T_2$, which must therefore be a bilinear function.

(ii) Suppose that T is the composite of the functions $U_1, U_2, \ldots, U_n$ drawn from our list of four elementary functions, so that

$$T = U_1 \circ U_2 \circ \cdots \circ U_n.$$

Now each of the U's is one-one, so the inverse function is

$$T^{-1} = U_n^{-1} \circ U_{n-1}^{-1} \circ \cdots \circ U_2^{-1} \circ U_1^{-1}.$$

If U is one of the four elementary functions, then so also is U^{-1}, so that T^{-1} is the composite of functions chosen from our list of four, and must therefore be a bilinear function. ■

(*continued from page 33*)

Consider now the image of a horizontal strip parallel to the x-axis in the z-plane, specified by

$$\{z : y \in [y_1, y_2], 0 \leqslant y_1 < y_2 < 2\pi\}.$$

If we choose the value of ϕ to be $\operatorname{Arg} w$, i.e. $\phi \in [0, 2\pi[$, then it follows from

$$e^{i\phi} = e^{iy}$$

that

$$\phi = y.$$

Since

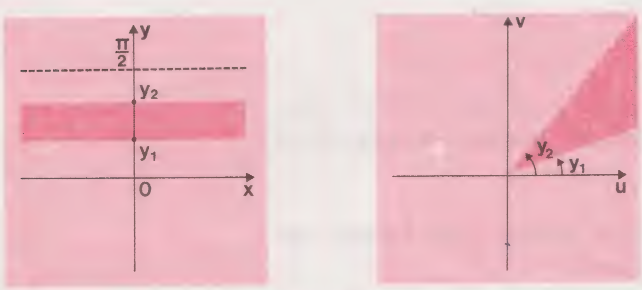
$$y \in [y_1, y_2],$$

we have

$$\phi \in [y_1, y_2],$$

which is a restriction on the values of w corresponding to the horizontal strip in the z-plane.

The set $\{w : \phi \in [y_1, y_2]\}$ is a wedge shape with its apex at the origin and apex angle $y_2 - y_1$.

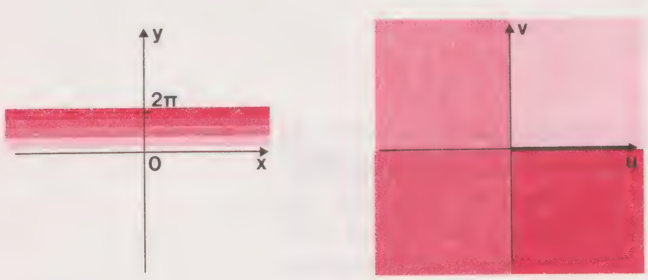


The exponential function
$z \mapsto e^z$

domain

codomain

If we allow the horizontal strip to be of width 2π (excluding 2π but including 0) then it will map to the complete w-plane.



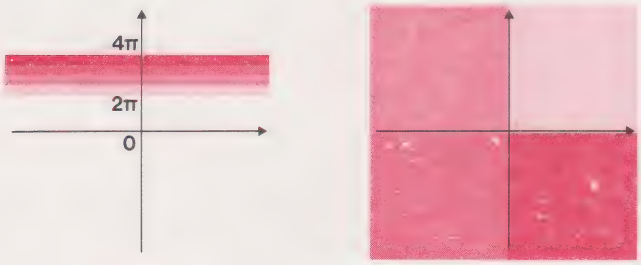
domain

codomain

If we map the horizontal strip

$$\{z : y \in [2\pi, 4\pi[\}$$

(also of width 2π) then the image is again the complete w-plane.



domain codomain

It is also interesting to see what happens to a strip parallel to the y-axis under the exponential function. Suppose that we take the set

$$\{z : x \in [x_1, x_2]\}$$

in the domain, then, again using polar co-ordinates (ρ, ϕ) in the codomain, we have

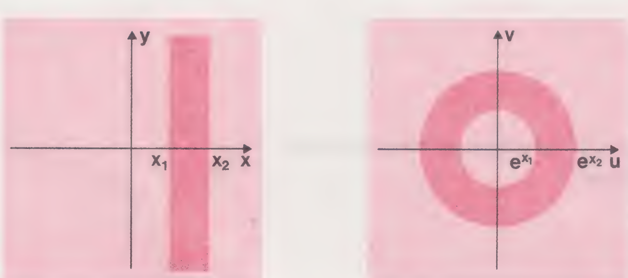
$$\rho = e^x,$$

and the corresponding set in the w-plane is

$$\{w : \rho \in [e^{x_1}, e^{x_2}]\}.$$

(Notice that we are using the fact that if $x_2 > x_1$ then $e^{x_2} > e^{x_1}$: if in doubt, see the graph of the (real) exponential function on page 51 of *Unit 7, Sequences and Limits I*.)

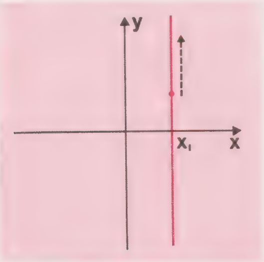
Since $\rho = |w|$, we see that the image is the annulus lying between the circles centred at the origin, with radii e^{x_1} and e^{x_2}.



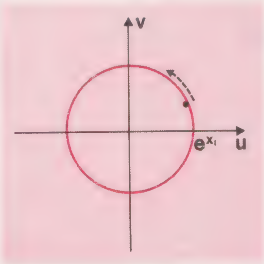
domain codomain

This is not, however, the complete story: although we have found the image of the set $\{z : x \in [x_1, x_2]\}$, there is one other important consideration. Suppose that a point on the line $x = x_1$ starts on the x-axis and moves up the line (i.e. in the direction of the positive y-axis). What is the locus of the image point in the w-plane, and how do the corresponding points move?

We can easily see from our previous discussion that the locus in the w-plane is simply the circle $\rho = |w| = e^{x_1}$; but how do the corresponding points move around this circle?



domain



codomain

As y increases from 0 to 2π, so $\phi = \operatorname{Arg} w = y$ increases from 0 to 2π, and the image point travels all the way round the circle in the w-plane. In fact, the image point w makes a complete circuit of the circle for each interval of length 2π on the line.

The function $z \longmapsto \exp z$ is many-one with a vengeance; an infinite number of points in the domain map to each point of the codomain.

Exercise 1

Exercise 1
(4 minutes)

(i) Find the set of elements which map to e^{x_1}, $x_1 \in R$, under the function

$$z \longmapsto \exp z.$$

(ii) Find the set of elements which map to $e^{x_1}e^{i\phi}$ under the function

$$z \longmapsto \exp z.$$ ■

Solution 1

Solution 1

(i) If $z = x + iy$, then

$$e^{x+iy} = e^{x_1},$$

so that

$$e^x e^{iy} = e^{x_1}(\cos 2n\pi + i \sin 2n\pi).$$

It follows that

$$x = x_1$$

and

$$y = 2n\pi \qquad (n \in Z).$$

(ii) If $z = x + iy$, then

$$e^x e^{iy} = e^{x_1} e^{i\phi},$$

so that

$$x = x_1$$

and

$$y = \phi + 2n\pi \qquad (n \in Z).$$ ■

29.4.2 The Joukowski Function

29.4.2

Discussion
* *

The function

$$z \longmapsto z + \frac{1}{z} \qquad (z \in C, z \neq 0)$$

is one of the classic functions of complex analysis and is associated by most mathematicians with the well known Joukowski aerofoil. In this text we only have time to discuss the simple example of a circle centred at the origin mapped to an ellipse, but some other shapes which can be produced from circles in the domain are shown in the television programme and reproduced in the television commentary. You can also find a nice discussion of this function in the book by Budden (see Bibliography) if you wish to pursue the subject further.

Every circle centred at the origin maps to an ellipse under the Joukowski function, with the exception of the interesting case of the circle with radius 1. (See Exercise 1.) This property of the Joukowski function is mentioned in the television programme, in connection with the design of aircraft.

Consider the circle

$$\{z : |z| = 2\}$$

in the domain, and, as before, let $w = u + iv$ be the image of z, so that

$$w = z + \frac{1}{z}.$$

On the circle $\{z : |z| = 2\}$ we have

$$z = 2e^{i\theta},$$

and therefore

$$\frac{1}{z} = \tfrac{1}{2}e^{-i\theta}.$$

It follows that

$$\begin{aligned} w &= z + \frac{1}{z} \\ &= 2e^{i\theta} + \tfrac{1}{2}e^{-i\theta} \\ &= 2(\cos\theta + i\sin\theta) + \tfrac{1}{2}(\cos\theta - i\sin\theta) \\ &= (2 + \tfrac{1}{2})\cos\theta + i(2 - \tfrac{1}{2})\sin\theta. \end{aligned}$$

We can now see that

$$u = \tfrac{5}{2}\cos\theta$$

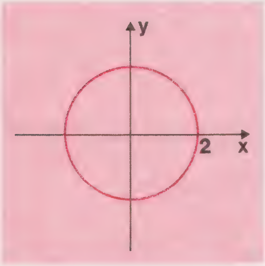
and

$$v = \tfrac{3}{2}\sin\theta,$$

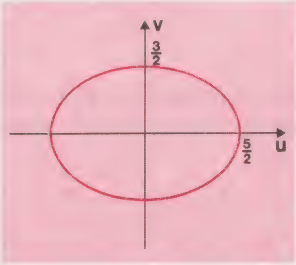
and hence that

$$\frac{4u^2}{25} + \frac{4v^2}{9} = 1,$$

which is the equation of an ellipse with semi-major and semi-minor axes of lengths $\frac{5}{2}$ and $\frac{3}{2}$ respectively.



domain



codomain

Exercise 1

Exercise 1
(3 minutes)

What is the image of the circle

$$\{z : |z| = 1\}$$

under the Joukowski function? ■

Solution 1

Solution 1

On the circle $\{z:|z| = 1\}$ we have

$$z = e^{i\theta},$$

and therefore

$$\frac{1}{z} = e^{-i\theta}.$$

It follows that (in the usual notation)

$$\begin{aligned} w &= e^{i\theta} + e^{-i\theta} \\ &= (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) \\ &= 2\cos\theta. \end{aligned}$$

Thus

$$u = 2\cos\theta \quad \text{and} \quad v = 0.$$

As

θ increases from 0 to π,

u decreases from 2 to -2;

as

θ increases from π to 2π,

u increases from -2 to 2.

We see that the image of $\{z:|z| = 1\}$ is the real interval $[-2, 2]$. (w traverses this interval twice — once in each direction — as z travels round the circle once.) ■

29.4.3 The "Square" Function

We have already mentioned some of the properties of the representation of the "square" function in section 29.2.1, but in this section we wish to examine it again in more detail, in preparation for the final section of this text.

Let us begin by attempting to find the image of the circle $\{z : |z| = 1\}$ under the "square" function

$$z \longmapsto z^2 \qquad (z \in C).$$

The restriction $|z| = 1$ on the values of z in the domain is equivalent to the restriction

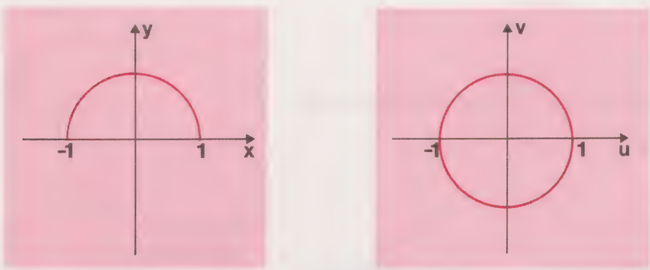
$$|z|^2 = 1 \quad \text{or} \quad |z^2| = 1.$$

If we let $w = z^2$ then the corresponding restriction on the values of w is

$$|w| = 1.$$

The image set is, therefore, the circle $\{w : |w| = 1\}$.

This is not, however, the whole story. We have already seen in section 29.2.1 that the semi-circle in the upper half of the domain also maps to this circle in the codomain.



domain codomain

This state of affairs may strike you as rather odd, and indeed it is. It all stems from the fact that the "square" mapping is many-one. We can see more clearly what is happening if we use polar co-ordinates.

We know that the "square" function can be represented in polar co-ordinates by

$$(r, \theta) \longmapsto (r^2, 2\theta) \qquad ((r, \theta) \in R_0^+ \times R),$$

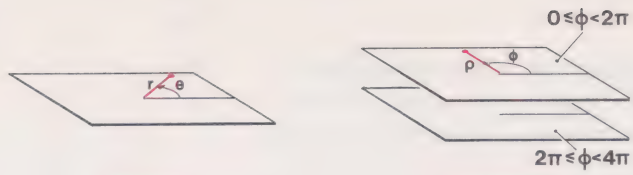
so that the argument of each point in the domain is doubled in the co-domain.

As z moves clockwise round the circle in the domain w moves twice round the circle in the codomain, also in a clockwise direction.

In order to get the full flavour of the function in this case we examine the behaviour of the corresponding polar co-ordinates. If we let (ρ, ϕ) be polar co-ordinates in the codomain, then

$$(\rho, \phi) = (r^2, 2\theta).$$

If we restrict θ to lie between 0 and 2π, it will imply that ϕ lies between 0 and 4π. We can show this behaviour quite clearly if we assign one sheet for the polar co-ordinates of the domain and two sheets for the polar co-ordinates of the codomain.



As a point traces out the semi-circle in the upper half of the domain we obtain a circle on the top sheet of the codomain.

If the point continues round the semi-circle in the lower half of the domain we obtain a circle on the lower sheet of the codomain.

This representation will have advantages when we discuss some reverse mappings in the next section.

Exercise 1

Exercise 1
(3 minutes)

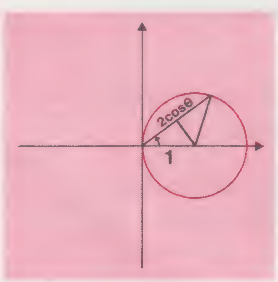
Show that the image of the line $x = a$ $(a > 0)$ under the "square" function

$$z \longmapsto z^2 \qquad (z \in C)$$

is a parabola. If a point moves up this line in the direction of the positive y-axis, how do the image points move? ■

Discussion
* *

There is a further interesting property of the "square" function, and that is the effect which it has on curves in the domain which pass through the origin. For example, consider the circle $\{z : |z - 1| = 1\}$ with centre $(1, 0)$ and radius 1.



It is best in this case to specify the circle in terms of polar co-ordinates, and on the circle we have

$$\left\{(r, \theta) : r = 2\cos\theta,\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}.$$

This restriction on the values of r and θ is identical to

$$r^2 = 4\cos^2\theta \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

If we let (ρ, ϕ) be polar co-ordinates in the codomain, then

$$\rho = r^2 = 4\cos^2\theta$$

and

$$\phi = 2\theta.$$

If we eliminate θ between these two equations, we obtain the corresponding restriction on the values of ρ and ϕ:

$$\rho = 4\cos^2\left(\frac{\phi}{2}\right)$$

which simplifies to (See RB10)

$$\rho = 2(1 + \cos\phi).$$

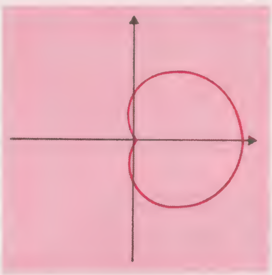
In addition, we had the restriction $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and since $\theta = \frac{\phi}{2}$, this becomes $\phi \in [-\pi, \pi]$. The image of the circle is therefore the set of complex numbers with polar co-ordinates lying in the set

$$\{(\rho, \phi) : \rho = 2(1 + \cos\phi), \phi \in [-\pi, \pi]\}.$$

The set with the equation

$$\rho = 2(1 + \cos\phi)$$

is called a *cardioid*.



Exercise 2

Exercise 2
(3 minutes)

As a point moves round the circle in the domain, how does the corresponding point move round the cardioid?

(Does it move twice round the cardioid?) ■

Solution 1

Solution 1

Let $w = u + iv$ be the variable in the codomain; then

$$w = z^2$$

and

$$u + iv = (x + iy)^2$$
$$= x^2 - y^2 + 2ixy,$$

so that

$$u = x^2 - y^2,$$

and

$$v = 2xy.$$

On the straight line in the domain we have $x = a$, hence

$$u = a^2 - y^2$$

and

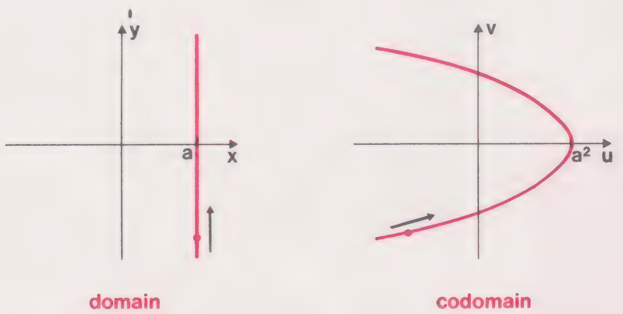
$$v = 2ay.$$

If we eliminate y between these equations we obtain

$$u = a^2 - \frac{v^2}{4a^2},$$

so that the image set is the parabola

$$\{w : v^2 = 4a^2(a^2 - u)\}.$$

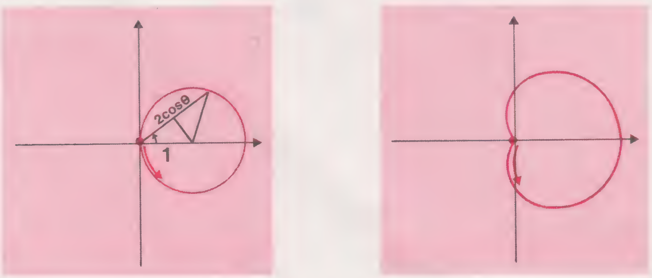


When y is large and negative, then u and v are both large and negative. As the point in the domain moves up the line so the image point moves along the parabola from the lower left, through the vertex and off to the upper left.

The image point crosses the imaginary axis as the point in the domain passes $(a, -a)$ and (a, a), and it crosses the real axis at the vertex of the parabola as the point in the domain crosses the real axis. ■

Solution 2

Solution 2



Since $\phi = 2\theta$, we know that as θ changes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, ϕ changes from $-\pi$ to π. The image points therefore traverse the cardioid once in the same direction. ■

29.5 THE *n*th ROOT MAPPING

29.5

29.5.1 Square Roots

29.5.1

Main Text
* * *

In *Unit 27, Complex Numbers I* we were interested in finding the solutions of the equation

$$z^2 = -1,$$

and we discovered that i and $-i$ are the "square roots" of -1. But how do we find the square roots of an arbitrary complex number?

Let us try the most obvious approach. Suppose that we are given a complex number $z_0 = x_0 + iy_0$ and we wish to find its square roots. We could attempt to solve the equation

$$z^2 = z_0,$$

that is,

$$(x + iy)^2 = x_0 + iy_0,$$

or

$$x^2 - y^2 + 2ixy = x_0 + iy_0.$$

Equating the real and imaginary parts, we obtain the simultaneous equations

$$x^2 - y^2 = x_0$$
$$2xy = y_0.$$

We can determine x from the second equation, that is

$$x = \frac{y_0}{2y}.$$

Then substituting in the first equation we get

$$\frac{y_0^2}{4y^2} - y^2 = x_0$$

i.e.

$$4y^4 + 4x_0y^2 - y_0^2 = 0.$$

This is a quadratic equation in y^2, which we can solve easily, and it will in general give rise to *four** values of y, and there will be four corresponding values of x. The end result is four number pairs as contenders for the square roots of z_0.

It seems that this technique is not only inelegant but it also produces *four* numbers of which we have to reject *two*. The two extra "solutions" are introduced when we square the expression for x in terms of y.

The conclusion surely is that the obvious approach is not very good and we should try something better. This would be even more apparent if we tried to find cube-roots this way.

The answer is to return to our definition of multiplication as an operation of scaling and rotation. Suppose that z_0 has polar co-ordinates (r, θ);

* Using the formula for the solution of a quadratic equation, we get

$$2y^2 = -x_0 \pm \sqrt{x_0^2 + y_0^2},$$

and whatever x_0, since $\sqrt{x_0^2 + y_0^2} \geqslant \sqrt{x_0^2} = |x_0|$, the apparent solution

$$2y^2 = -x_0 - \sqrt{x_0^2 + y_0^2} \leqslant 0$$

is no solution since y is real and so y^2 cannot be negative.

then we wish to find a complex number w_0 with polar co-ordinates (ρ, ϕ), say, such that

$$w_0^2 = z_0.$$

The polar co-ordinates of w_0^2 are $(\rho^2, 2\phi)$ and therefore

$$(\rho^2, 2\phi) = (r, \theta).$$

We might conclude that*

$$\rho = \sqrt{r} \quad \text{and} \quad \phi = \frac{\theta}{2},$$

and therefore the complex number with polar co-ordinates $\left(\sqrt{r}, \frac{\theta}{2}\right)$ is a square root of z_0.

We have obtained only one square root, but after the next exercise we shall discuss a simple way of getting the second.

Exercise 1

Exercise 1
(3 minutes)

Find polar co-ordinates for the complex number $1 + i\sqrt{3}$, and hence find one of its square roots. ■

Main Text

The reason why we only get the one square root is that we inferred from the equation

$$w_0^2 = z_0$$

that

$$(\rho^2, 2\phi) = (r, \theta),$$

which is quite unjustified. We know that the polar co-ordinate representation of a complex number is *not unique*, and, before we can write down such an equation, we must at least verify that θ is chosen so that the values of ϕ obtained cover all solutions for w_0. In fact, the non-uniqueness, which has only been a nuisance up till now, can be put to good use in this context.

Suppose $\theta = \text{Arg}\, z_0$; then one solution of $w_0^2 = z_0$ can be obtained from

$$(\rho^2, 2\phi) = (r, \theta).$$

But we could just as well choose $\theta + 2k\pi$ $(k \in Z)$, instead of θ. And if, for instance, we choose $\theta + 2\pi$, we get

$$(\rho^2, 2\phi) = (r, \theta + 2\pi)$$

whence

$$\phi = \frac{\theta}{2} + \pi.$$

Now we have two solutions

$$\left(\sqrt{r}, \frac{\theta}{2}\right) \quad \text{and} \quad \left(\sqrt{r}, \frac{\theta}{2} + \pi\right)$$

and these represent *different* complex numbers. Let us just have a look at this in terms of the numerical example in the previous exercise.

In that exercise, it is true that $1 + i\sqrt{3}$ has polar co-ordinates $\left(2, \frac{\pi}{3}\right)$ but it also has polar co-ordinates $\left(2, \frac{7\pi}{3}\right)$, for instance. In fact, the elements

* When we write $\rho = \sqrt{r}$ we mean the positive square root, since $\rho = |w_0|$ is positive.

of any one of the pairs

$$\left(2, \frac{\pi}{3} + 2k\pi\right) \qquad (k \in Z)$$

are polar co-ordinates of the complex number $1 + i\sqrt{3}$.

If we now attempt to find the square roots, we find that their polar co-ordinates are

$$\left(\sqrt{2}, \frac{\pi}{6} + k\pi\right) \qquad (k \in Z).$$

At first sight it seems that we have swung too far in the opposite direction; instead of just one square root we now appear to have an infinite number. Luckily, all is well, for if we find the corresponding complex numbers, we obtain the set of elements of the form

$$\sqrt{2}\cos\left(\frac{\pi}{6} + k\pi\right) + i\sqrt{2}\sin\left(\frac{\pi}{6} + k\pi\right) \qquad (k \in Z),$$

and this set contains only *two* distinct elements, corresponding to $k = 0$ and $k = 1$ say:

$$\frac{\sqrt{3} + i}{\sqrt{2}} \quad \text{and} \quad \frac{-\sqrt{3} - i}{\sqrt{2}}.$$

It is not difficult to see why this is so. If, for instance, $k = 2$, then the angle becomes $\frac{\pi}{6} + 2\pi$ and this gives the same complex number as $k = 0$. In general, for any k, $k + 2$ and k give the same complex number.

Exercise 2
(3 minutes)

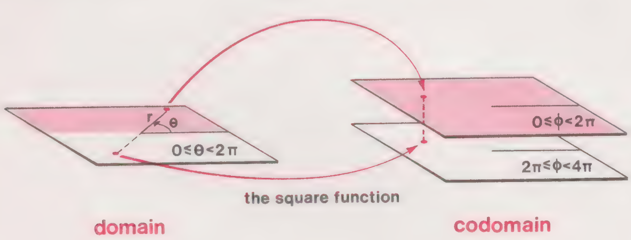
Exercise 2

Find both the square roots of $\sqrt{2} + i\sqrt{2}$. ■

Discussion
* * *

We shall call the reverse mapping of the "square" function the "square root" mapping. This mapping is *not* a function because it is one-many. In order to understand the behaviour of the "square root" mapping, it is helpful to recall what we discovered about the "square" function.

The "square" function maps the set $\{(r, \theta): 0 \leqslant \theta < 2\pi\}$ to the set $\{(r^2, 2\theta): 0 \leqslant \theta < 2\pi\}$.



If we now introduce the many-one mapping p from polar co-ordinates to the corresponding Cartesian co-ordinates:

$$p: (r, \theta) \longmapsto (x, y),$$

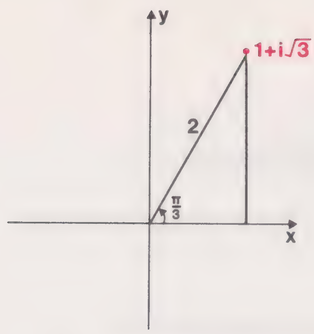
we have

$$\begin{array}{ccc} \textit{Domain} & & \textit{Codomain} \\ (x, y) & \xrightarrow{\text{square}} & (u, v) \\ \Big\downarrow {\scriptstyle \text{reverse } p} & & \Big\uparrow {\scriptstyle p} \\ (r, \theta + 2k\pi) & \xrightarrow{\text{square}} & (r^2, 2\theta + 4k\pi) \end{array}$$

(*continued on page 49*)

Solution 1

Solution 1



The complex number $1 + i\sqrt{3}$ has polar co-ordinates $\left(2, \frac{\pi}{3}\right)$; it follows that the complex number with polar co-ordinates $\left(\sqrt{2}, \frac{\pi}{6}\right)$ is a square-root of $1 + i\sqrt{3}$. If w_0 is this square root, then

$$\begin{aligned} w_0 &= \sqrt{2}\cos\frac{\pi}{6} + i\sqrt{2}\sin\frac{\pi}{6} \\ &= \sqrt{2} \times \frac{\sqrt{3}}{2} + i\sqrt{2} \times \frac{1}{2} \\ &= \frac{\sqrt{3} + i}{\sqrt{2}}. \end{aligned}$$

■

Solution 2

Solution 2

$\sqrt{2} + i\sqrt{2}$ has polar co-ordinates $\left(2, \frac{\pi}{4} + 2k\pi\right)$ $(k \in Z)$. The square roots of $\sqrt{2} + i\sqrt{2}$ therefore have polar co-ordinates

$$\left(\sqrt{2}, \frac{\pi}{8} + k\pi\right) \qquad (k \in Z),$$

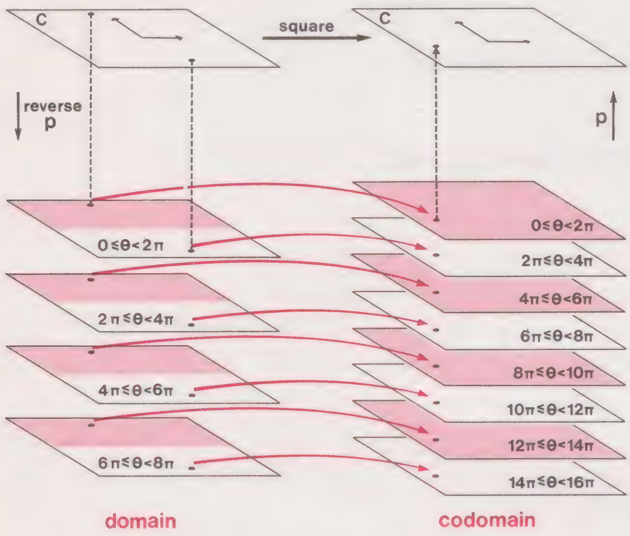
and there are just two different square roots:

$$\sqrt{2}e^{i(\pi/8)} \quad \text{and} \quad \sqrt{2}e^{9i\pi/8}.$$

■

(*continued from page 47*)

If we introduce *all* the values of the argument of a particular complex number, and use a different sheet for each interval $2k\pi \leqslant \theta < 2(k+1)\pi$, we have diagrammatically:



In order to find the square of a complex number using polar co-ordinates, we could perform the operations:

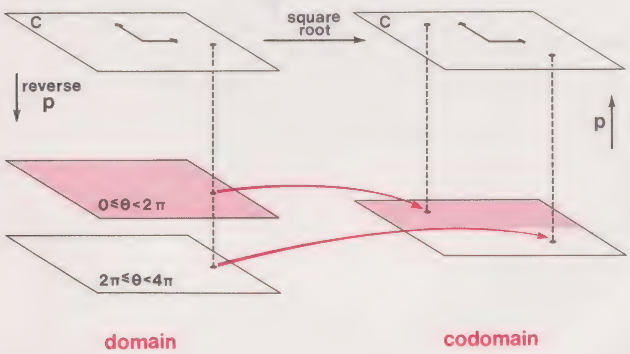
(i) express complex number in polar co-ordinates;
(ii) square it;
(iii) convert polar co-ordinates to a complex number.

In other words, we perform the composite mapping

$$p \circ (\text{square}) \circ (\text{reverse of } p).$$

From the diagram we can see that this process maps two points in the domain to one point in the codomain. In order to produce the image of a particular complex number we need only take values of θ lying on the sheet corresponding to $0 \leqslant \theta < 2\pi$. But think now what would happen if we were to proceed round this diagram in the reverse direction as we need to do for the "square root" mapping. In order to produce both square roots of a given complex number we need two adjacent sheets for the polar co-ordinates, for example, the two sheets corresponding to

$$0 \leqslant \theta < 2\pi \quad \text{and} \quad 2\pi \leqslant \theta < 4\pi.$$



The value of θ between 0 and 2π gives one root and the value of θ between 2π and 4π gives the other. We need not take any more of the possible values for θ, as they will simply repeat the roots which we have already found.

29.5.2 *n*th Roots

29.5.2

Main Text

In order to find the nth root of a given complex number, we need only follow the pattern which we have set for square roots. If the given complex number z_0 has $\operatorname{Arg} z_0 = \theta$ and modulus r, then z_0 is represented by $(r, \theta + 2k\pi)$ where $k \in Z$.

Then the nth roots of z_0 are the solutions w_0 of the equation

$$w_0^n = z_0.$$

If w_0 is represented by (ρ, ϕ) then w_0^n is represented by $(\rho^n, n\phi)$.

From

$$(\rho^n, n\phi) = (r, \theta + 2k\pi),$$

we get

$$\rho = r^{1/n}, \qquad \phi = \frac{\theta + 2k\pi}{n} \qquad (k \in Z),$$

whence

$$w_0 = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right).$$

Taking n *successive* values of k will produce the correct number of distinct nth roots: taking further values of k will just repeat the roots already found. It is convenient to take $k = 0, 1, \ldots, n - 1$.

Exercise 1

Exercise 1
(3 minutes)

(i) Find the values of $(1 + i)^{1/5}$.
(ii) Find the values of $(1 + i)^{2/5}$. ■

Exercise 2

Exercise 2
(5 minutes)

Solve the equation

$$z^5 = (1 - z)^5.$$ ■

Exercise 3 (Optional)

Exercise 3
(5 minutes)

Show that all the roots of the equation

$$(z + i)^5 = (z - i)^5$$

are real.

(HINT: There is a quick way!) ■

29.6 CONCLUSION

29.6

Main Text

In this text we have been concerned with the notion of a *complex function*. We have seen how we can extend the definition of the real exponential function to that of the complex exponential function; we have also discussed a few other special functions. In particular, we discussed the *bilinear function* and showed that the set of all circles and straight lines is invariant under a bilinear function. When we discussed the "square" function, we saw that the *upper half* of the z-plane corresponded to the *whole* of the w-plane, since the argument of each point in the domain is doubled in the codomain.

In section 29.5 we discussed the nth roots of a complex number. We began by discussing the "square root" mapping, which is the reverse of the "square" function. We saw that a complex number α has just two square roots, that is, an equation of the form

$$z^2 - \alpha = 0$$

has just two complex roots. We then considered the nth root mapping and saw that an equation of the form

$$z^n - \alpha = 0 \qquad (n \geqslant 1)$$

has just n complex roots.

There is an important theorem, known as the *Fundamental Theorem of Algebra*, which states that every polynomial equation of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 \qquad (n \geqslant 1),$$

where $a_0, a_1, \ldots, a_n \in C$ and $a_n \neq 0$, has at least one complex root. The proof of this theorem is beyond the scope of the Foundation Course, but if we assume this result, we can indicate a proof of the following theorem.

CONJECTURE

Every polynomial equation of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 \qquad n \geqslant 1,$$

where $a_0, a_1, \ldots, a_n \in C$ and $a_n \neq 0$, has at most n complex roots.

This result can be established as follows (see *Unit 17*, section 17.2.3).

"PROOF" BY MATHEMATICAL INDUCTION

FIRST STEP

Consider the case $n = 1$.

The equation

$$a_1 z + a_0 = 0 \qquad (a_1 \neq 0)$$

has only one complex root, namely

$$-\frac{a_0}{a_1}.$$

SECOND STEP

Assume that an equation of the form

$$b_k z^k + \cdots + b_1 z + b_0 = 0 \qquad (k \geqslant 1)$$

where $b_0, b_1, \ldots, b_k \in C$ and $b_k \neq 0$, has at most k complex roots.

Solution 1

Solution 1

(i) $$(1 + i) = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right) \qquad (k \in Z),$$

hence, using the formula in the text, the five values of $(1 + i)^{1/5}$ are

$$2^{1/10}\left(\cos\left(\frac{\pi}{20} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi}{20} + \frac{2k\pi}{5}\right)\right) \qquad (k = 0, 1, 2, 3, 4).$$

(ii) Similarly, the five values of $(1 + i)^{2/5}$ are

$$2^{1/5}\left(\cos\left(\frac{\pi}{10} + \frac{4k\pi}{5}\right) + i\sin\left(\frac{\pi}{10} + \frac{4k\pi}{5}\right)\right) \qquad (k = 0, 1, 2, 3, 4).$$

■

Solution 2

Solution 2

We need to solve the equation

$$\left(\frac{z}{1 - z}\right)^5 = 1 \qquad (z \neq 1).$$

Let

$$w = \frac{z}{1 - z},$$

then the roots of the equation $w^5 = 1$ are

$$1, e^{2i\pi/5}, e^{4i\pi/5}, e^{6i\pi/5}, e^{8i\pi/5}.$$

If w_1 is a root of $w^5 = 1$, then

$$w_1 = \frac{z_1}{1 - z_1},$$

where z_1 is a root of the original equation. Solving for z_1 we obtain

$$z_1 = \frac{w_1}{1 + w_1}.$$

The roots of the original equation are therefore

$$\frac{1}{2}, \frac{e^{2i(\pi/5)}}{1 + e^{2i(\pi/5)}}, \frac{e^{4i(\pi/5)}}{1 + e^{4i(\pi/5)}}, \frac{e^{6i(\pi/5)}}{1 + e^{6i(\pi/5)}}, \frac{e^{8i(\pi/5)}}{1 + e^{8i(\pi/5)}}.$$

■

Solution 3

Solution 3

If

$$(z + i)^5 = (z - i)^5$$

then

$$|z + i|^5 = |z - i|^5,$$

so that

$$|z + i| = |z - i|.$$

All solutions of the original equation are therefore at equal distances from the points corresponding to the complex numbers $-i$ and i; in other words, they lie on the real axis. ■

(continued from page 51)

Consider an equation of the form

$$a_{k+1}z^{k+1} + a_k z^k + \cdots + a_1 z + a_0 = 0 \qquad (k + 1 \geqslant 1),$$

where $a_0, a_1, \ldots, a_{k+1} \in C$ and $a_{k+1} \neq 0$. By the Fundamental Theorem of Algebra, this equation has at least one complex root, α say. This means that we can rewrite the equation in the form

$$(z - \alpha)(b_k z^k + \cdots + b_1 z + b_0) = 0,$$

where $k \geqslant 1$, $b_0, b_1, \ldots, b_k \in C$ and $b_k \neq 0$ (since $b_k = a_{k+1}$). (We have not *proved* this step, so we are indicating a proof of the conjecture, rather than actually giving one.)

Now, by conjecture, the equation

$$b_k z^k + \cdots + b_1 z + b_0 = 0$$

has at most k complex roots, and it follows that the equation

$$a_{k+1}z^{k+1} + a_k z^k + \cdots + a_1 z + a_0 = 0$$

has at most $k + 1$ complex roots.

That is, if the conjecture is TRUE for k ($k \in Z^+$), then it is TRUE for $k + 1$. It is TRUE for $k = 1$, and hence it is TRUE for all positive integers n.

Our discussion of nth roots and the examples considered in section 29.5 provide illustrations of this important theorem.

There is a vast theory of complex functions, just as there is for real functions, and we have barely touched the fringes of it. However, we hope that we have enabled you to glimpse some of the intriguing possibilities of a subject which must rank as one of the most elegant areas of mathematics.

Notes

Notes

M100 – MATHEMATICS FOUNDATION COURSE UNITS

1 Functions
2 Errors and Accuracy
3 Operations and Morphisms
4 Finite Differences
5 NO TEXT
6 Inequalities
7 Sequences and Limits I
8 Computing I
9 Integration I
10 NO TEXT
11 Logic I — Boolean Algebra
12 Differentiation I
13 Integration II
14 Sequences and Limits II
15 Differentiation II
16 Probability and Statistics I
17 Logic II — Proof
18 Probability and Statistics II
19 Relations
20 Computing II
21 Probability and Statistics III
22 Linear Algebra I
23 Linear Algebra II
24 Differential Equations I
25 NO TEXT
26 Linear Algebra III
27 Complex Numbers I
28 Linear Algebra IV
29 Complex Numbers II
30 Groups I
31 Differential Equations II
32 NO TEXT
33 Groups II
34 Number Systems
35 Topology
36 Mathematical Structures